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Exactly solvable models of scattering with $SL(2, \mathbf{C})$ symmetry

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Abstract

Using the theory of induced representations two exactly solvable models of non-relativistic scattering with $SL(2, \mathbf{C})$ symmetry are presented. The first describes the scattering of a charged particle moving on the Poincaré upper half space $\mathbf{H}$ under the influence of an $SU(2)$ non-Abelian gauge potential with isospin s. The second deals with a one-dimensional coupled-channel scattering problem for a charged particle in a matrix-valued scalar potential containing Morse-like interaction terms. The coupled channel wavefunctions and the corresponding scattering matrices are calculated. A detailed description of the underlying geometric structures is also given and a generalization for restricting the motion to fundamental domains of $\mathbf{H}$ (three manifolds of constant negative sectional curvature) is outlined. Such models provide an interesting generalization to the known ones of multichannel scattering, quantum chaos and chaotic cosmology.

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1. Introduction

Exactly solvable models of quantum-mechanical scattering related to an underlying non-compact symmetry group G generated considerable interest in the past few decades. The advent of algebraic scattering theory (AST) [1] revealed that group theoretical methods can be extended successfully from the investigation of bound state problems to the scattering region as well. Group theoretical methods (using both discrete and continuous symmetry groups) facilitate explicit construction of scattering wavefunctions, and the corresponding scattering matrices as needed for model building in nuclear physics [2], molecular and atomic physics [3] and in the description of quantum chaos [4, 5].

The simplest and best studied examples were based on the symmetry groups $SO(2, 1)$ and $SO(3, 1)$, the proper orthochronous Lorentz groups, or their covering groups $SL(2, \mathbf{R})$ and

$SL(2, \mathbf{C})$. The corresponding Lorentz groups are related to the dynamical symmetry group of the non-relativistic Coulomb problem in two and three dimensions [6], and to a class of exactly solvable scattering problems with Pöschl–Teller and Morse potentials. The corresponding covering groups were used first in the investigations of quantum chaos. The Poincaré upper half-plane model with an $SL(2, \mathbf{R})$ symmetry restricted to a suitable fundamental domain has been studied with [5] and without [4] inclusion of a constant magnetic field and with Aharonov–Bohm fluxes [7]. That is now the archetypal example of chaotic quantum scattering in multiply connected spaces. The inclusion of Aharonov–Bohm fluxes to such models is also a useful concept for the models of persistent currents in mesoscopic systems in solid-state physics [8]. As far as the author knows, models describing scattering situations with $SL(2, \mathbf{C})$ symmetry using the Poincaré upper half space $\mathbf{H}$ have received little attention to date. Models with $SL(2, \mathbf{C})$ symmetry suitably restricted to fundamental domains of $\mathbf{H}$ so far have been used only in connection with bound state problems. In this respect, note that in [12] and [13], the possible occurrence of quantum chaos in Robertson–Walker cosmologies has been considered.

The aim of this study is then to display an interesting class of exactly solvable models with $SL(2, \mathbf{C})$ symmetry that describe physically interesting scattering problems. Two model systems are considered. The first describes the quantum-mechanical scattering of a charged particle moving on the Poincaré upper half space $\mathbf{H}$ under the influence of an $SU(2)$ non-Abelian gauge potential with the isospin s. This model is the natural generalization of the corresponding motion on the upper half plane under the influence of an Abelian $U(1)$ gauge field producing a constant magnetic field as studied by Comtet *et al* [5, 9, 10]. The second deals with a one-dimensional coupled-channel scattering problem for a charged particle in a matrix-valued scalar potential containing Morse-like interaction terms. Since the seminal work of Morse [11], it is well known that the bound states of (single channel) Morse-like interaction terms are useful for treating the vibrational motion of a diatomic molecule. As far as the scattering states are concerned a systematic study based on an underlying symmetry algebra has not appeared yet. Our second model is an example of that kind. It is an analytically solvable multichannel scattering problem with $SL(2, \mathbf{C})$ symmetry. For these problems the coupled channel wavefunctions and the corresponding scattering matrices can be explicitly calculated. Moreover, by using suitable discrete subgroups Γ of $SL(2, \mathbf{C})$, a large class of solvable models describing scattering problems on three manifolds of the form $SL(2, \mathbf{C})/\Gamma$ can be identified as well. These models are the higher dimensional analogues of those describing chaotic scattering under the influence of a 'constant' non-Abelian $SU(2)$ gauge field. Such models provide an interesting generalization to what is used currently in the studies of quantum chaos and chaotic cosmology.

The organization of this paper is as follows. In section 2, I present the matrix-valued $SL(2, \mathbf{C})$ realization and clarify the associated underlying geometrical structures. In section 3, the corresponding Casimir operators are specified and their relationships to the non-relativistic Hamiltonians defining solvable scattering problems are established. Next, in section 4, a convenient ansatz for the separation of variables is defined and the group theoretical meaning of the ensuing coupled channel wavefunction is clarified. To gain insight for the matrix-valued radial problem for an arbitrary isospin s, the spin-$\frac{1}{2}$ and spin-1 cases are worked through explicitly in section 5 while the solution of the radial problem for the general isospin s is given in section 6. The explicit forms of the scattering matrix and of the coupled channel wavefunction are given therein. Following, in section 7, I outline a method for generalizing the models for three manifolds of the form $SL(2, \mathbf{C})/\Gamma$. Finally, in section 8, I make some observations on the possible use of these notions in the context of algebraic scattering theory, of quantum chaos and of chaotic cosmology along with conclusions from this work. Some calculation details can be found in the three appendices.

2. An *SL*(2, C) realization

Let the upper half space $\mathbf{H}$ be the set

$$\mathbf{H} \equiv \{(x, y, t) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^+\} = \{(z, t) | x + \mathrm{i}y \in \mathbf{C}, t \in \mathbf{R}, t > 0\} \tag{1}$$

and it will also be useful to use the notation x^μ, $\mu = 1, 2, 3$, for the coordinates (x, y, t), i.e. $(x, y, t) \equiv (x^1, x^2, x^3)$. This $\mathbf{H}$ can be identified [14] with the coset $SL(2, \mathbf{C})/SU(2)$. Due to this identification there is a natural action of $SL(2, \mathbf{C})$ on $\mathbf{H}$. As is well known there is a hyperbolic metric on $\mathbf{H}$, the Poincaré metric, defined by the line element

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}t^2}{t^2} = \frac{(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2}{(x^3)^2} = g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu. \tag{2}$$

$SL(2, \mathbf{C})$ acts on $\mathbf{H}$ as an isometry of this metric with the following set of six Killing vectors:

$$G_1 = -\mathrm{i}\left(y(x\partial_x + t\partial_t) + \tfrac{1}{2}(1 - x^2 + y^2 - t^2)\partial_y\right) \tag{3}$$

$$G_2 = \mathrm{i}\left(x(y\partial_y + t\partial_t) + \tfrac{1}{2}(1 + x^2 - y^2 - t^2)\partial_x\right) \tag{4}$$

$$G_3 = -\mathrm{i}(x\partial_y - y\partial_x) \tag{5}$$

$$F_1 = \mathrm{i}\left(x(y\partial_y + t\partial_t) - \tfrac{1}{2}(1 - x^2 + y^2 + t^2)\partial_x\right) \tag{6}$$

$$F_2 = \mathrm{i}\left(y(x\partial_x + t\partial_t) - \tfrac{1}{2}(1 + x^2 - y^2 + t^2)\partial_y\right) \tag{7}$$

$$F_3 = -\mathrm{i}(x\partial_x + y\partial_y + t\partial_t) \tag{8}$$

which satisfy the commutation relations of the $sl(2, \mathbf{C})$ algebra,

$$[G_k, G_l] = \mathrm{i}\epsilon_{klm}G_m \quad [G_k, F_l] = \mathrm{i}\epsilon_{klm}F_m \quad [F_k, F_l] = -\mathrm{i}\epsilon_{klm}G_m \quad j, k, l = 1, 2, 3. \tag{9}$$

Note that the ranges of the coordinate indices μ, ν and of the Lie-algebra indices j, k, l are the same. They all may take the values 1, 2 and 3.

Using the theory of induced representations, suitable matrix-valued modifications can be made to these Killing vectors still leaving the $sl(2, \mathbf{C})$ commutation relations intact. Since $\mathbf{H} \simeq SL(2, \mathbf{C})/SU(2)$, this can be achieved by choosing an irreducible unitary representation for $SU(2)$ characterized by a particular value of the spin s. If the generators of this unitary irreducible representation are denoted by S_j, they are $(2s + 1) \times (2s + 1)$-matrices satisfying the usual relations $[S_j, S_k] = \mathrm{i}\epsilon_{jkl}S_l$. Then the generators of the induced representation for $SL(2, \mathbf{C})$ induced by this unirep of $SU(2)$ are matrix-valued differential operators [15]. In mathematics, they are operators acting on wavefunctions that are the sections of the $(2s + 1)$-dimensional vector bundle over $SL(2, \mathbf{C})/SU(2)$, an associated bundle to the canonical bundle with total space $SL(2, \mathbf{C})$) over the base space $SL(2, \mathbf{C})/SU(2)$ with fibre $SU(2)$.

These $sl(2, \mathbf{C})$ generators of the representation induced by the spin s representation of $su(2)$ are

$$\mathcal{G}_1 = G_1 + tS_1 - xS_3 \qquad \mathcal{G}_2 = G_2 + tS_2 - yS_3 \qquad \mathcal{G}_3 = G_3 + S_3 \tag{10}$$

$$\mathcal{F}_1 = F_1 + tS_2 - yS_3 \qquad \mathcal{F}_2 = F_2 + xS_3 - tS_1 \qquad \mathcal{F}_3 = F_3. \tag{11}$$

This new set of generators satisfies the set of commutation relations

$$[\mathcal{G}_k, \mathcal{G}_l] = \mathrm{i}\epsilon_{klm}\mathcal{G}_m \quad [\mathcal{G}_k, \mathcal{F}_l] = \mathrm{i}\epsilon_{klm}\mathcal{F}_m \quad [\mathcal{F}_k, \mathcal{F}_l] = -\mathrm{i}\epsilon_{klm}\mathcal{G}_m \quad j, k, l = 1, 2, 3 \tag{12}$$

and so gives a realization of the $sl(2, \mathbf{C})$ algebra in terms of matrix-valued differential operators. To render the paper self-contained, an explicit construction for both the sets of generators (G_j, F_k) and $(\mathcal{G}_j, \mathcal{F}_k)$ by using previous results [15] is given in appendix A.

3. Casimir operators and Hamiltonians

As $SL(2,\mathbf{C})$ is a group of rank two, there are two independent Casimir operators. For realizations given in equations (3)–(8), (10) and (11), they are

$$C_1 = G^2 - F^2 \qquad C_2 = G_j F_j = F_j G_j \tag{13}$$

$$\mathcal{C}_1 = \mathcal{G}^2 - \mathcal{F}^2 \qquad \mathcal{C}_2 = \mathcal{G}_j \mathcal{F}_j = \mathcal{F}_j \mathcal{G}_j \tag{14}$$

in which it is understood that repeated indices are summed. Then, it can be shown that

$$C_1 = t^2\partial_t^2 - t\partial_t + t^2\left(\partial_x^2 + \partial_y^2\right) \qquad C_2 = 0 \tag{15}$$

and

$$\mathcal{C}_1 = C_1 + 2\mathrm{i}t(S_2\partial_x - S_1\partial_y) + S_3^2 \qquad \mathrm{i}\mathcal{C}_2 = t(S_1\partial_x + S_2\partial_y + S_3\partial_t) - S_3. \tag{16}$$

Next, it is useful to introduce

$$A = A_\mu^j S_j\,\mathrm{d}x^\mu = \frac{1}{x^3}(S_1\,\mathrm{d}x^2 - S_2\,\mathrm{d}x^1) = \frac{1}{t}(S_1\,\mathrm{d}y - S_2\,\mathrm{d}x) \qquad \nabla_\mu \equiv \partial_\mu - \mathrm{i}A_\mu. \tag{17}$$

The one-form A is an $SU(2)$-valued gauge field living on $\mathbf{H}$. Or, more precisely, A is the pull-back of the canonical connection to our vector bundle associated with the principal bundle $(SL(2,\mathbf{C}), SU(2), \mathbf{H})$ with respect to the section, equation (146) defined in appendix A. Here also ∇_μ is the usual covariant derivative containing the gauge field. Also, I introduce the generalization of the Laplace–Beltrami operator by replacing ∂_μ in the usual definition by ∇_μ. Using the definitions above and equation (2) gives

$$\triangle(A) \equiv \frac{1}{\sqrt{g}}\nabla_\mu(\sqrt{g}g^{\mu\nu}\nabla_\nu) = \mathcal{C}_1 - S^2 \tag{18}$$

so that the Landau-like Hamiltonian $H = -\triangle(A)$ ($\hbar = 2m = 1$) is the difference of the quadratic Casimir operators of $SL(2,\mathbf{C})$ and $SU(2)$ in the induced and inducing representations, respectively. This result is a special case of a more general result well known in the literature (see [15], and references therein). One can also calculate the field-strength (curvature two-form) $F = \mathrm{d}F - \mathrm{i}A \wedge A$ which is

$$F = \frac{1}{2}F_{\mu\nu}^j S_j\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2t^2}\epsilon_{\mu\nu j}S_j\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho}B_\rho^j S_j\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{19}$$

where the last entry in equation (19) defines the 'magnetic field' $\mathbf{B}$. In vector notation with matrix indices being implicit,

$$\mathbf{B} = \frac{1}{t^2}\mathbf{S}. \tag{20}$$

Note that the 'magnetic field' of equation (20) is a natural generalization of the single-component quantity (the 'constant magnetic field') introduced in the context of the upper half *plane* model $SL(2,\mathbf{R})/U(1)$ with the line element $\mathrm{d}s^2 = \frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$ via the two-form $F = B\frac{\mathrm{d}x\wedge\mathrm{d}y}{y^2}$ [5]. Equation (20) also reveals that, in this non-Abelian generalization, the $su(2)$ generators $\mathbf{S}$ play the role of the constant magnetic field B of the upper half *plane* model. The Hamiltonian $H \equiv -\triangle(A)$ defines the first model in this study. It describes the motion of a charged particle in $\mathbf{H}$ under the influence of an $SU(2)$ gauge field with the isospin s.

However, there is also a related one-dimensional solvable scattering problem. This can be obtained by using a similarity transformation $S(t) \equiv t$ and a coordinate transformation $t = \mathrm{e}^{-r}$ on the Casimir operators giving

$$\mathcal{C}_1' \equiv S^{-1}\mathcal{C}_1 S = \partial_r^2 + \mathrm{e}^{-2r}\triangle + 2\,\mathrm{i}\,\mathrm{e}^{-r}(S_2\partial_x - S_1\partial_y) + S_3^2 - 1 \tag{21}$$

$$\mathrm{i}\mathcal{C}_2' \equiv S^{-1}(\mathrm{i}\mathcal{C}_2)S = -S_3\partial_r + \mathrm{e}^{-r}(S_1\partial_x + S_2\partial_y) \tag{22}$$

where $\triangle = \partial_x^2 + \partial_y^2$ is the Laplace operator on $\mathbf{R}^2$. An alternative form of these operators can be found by identifying $\mathbf{R}^2$ with $\mathbf{C}$, introducing complex coordinates $z = x + \mathrm{i}y$ and $\overline{z} = x - \mathrm{i}y$, and having the basis for one-forms and vector fields as

$$\mathrm{d}z = \mathrm{d}x + \mathrm{i}\,\mathrm{d}y \quad \mathrm{d}\overline{z} = \mathrm{d}x - \mathrm{i}\,\mathrm{d}y \quad \overline{\partial} = \tfrac{1}{2}(\partial_x + \mathrm{i}\partial_y) \quad \partial = \tfrac{1}{2}(\partial_x - \mathrm{i}\partial_y) \tag{23}$$

satisfying $\mathrm{d}z(\partial) = \mathrm{d}\overline{z}(\overline{\partial}) = 1$, $\mathrm{d}z(\overline{\partial}) = \mathrm{d}\overline{z}(\partial) = 0$. Note that the Laplacian on $\mathbf{C}$ in these coordinates has the form $\triangle = 4\partial\overline{\partial}$. The operators ∂ and $\overline{\partial}$ are the usual Cauchy–Riemann operators with complex variables. In these coordinates the Casimir operators become

$$\mathcal{C}_1' = \partial_r^2 + \mathrm{e}^{-2r}\triangle + 2\,\mathrm{e}^{-r}(S_+\partial - S_-\overline{\partial}) + S_3^2 - 1 \tag{24}$$

$$\mathrm{i}\mathcal{C}_2' = -S_3\partial_r + \mathrm{e}^{-r}(S_+\partial + S_-\overline{\partial}) \tag{25}$$

where $S_\pm = S_1 \pm \mathrm{i}S_2$.

Since the Laplace operator $\triangle = 4\overline{\partial}\partial$ commutes with $\mathcal{C}_1'$ and $\mathcal{C}_2'$, one can define new operators as

$$\mathcal{S}_+ = \mathcal{S}_1 + \mathrm{i}\mathcal{S}_2 = S_+ \otimes D \quad \mathcal{S}_- = \mathcal{S}_1 - \mathrm{i}\mathcal{S}_2 = S_- \otimes D^\dagger \quad \mathcal{S}_3 = S_3 \otimes \mathbf{1} \tag{26}$$

where

$$D \equiv \frac{2}{\mathrm{i}\lambda}\partial \qquad D^\dagger \equiv \frac{2}{\mathrm{i}\lambda}\overline{\partial} \tag{27}$$

acting on $\mathbf{C}^{2s+1} \otimes \mathcal{F}^\lambda$. They form the space of complex, vector-valued functions satisfying $\left(\triangle + \lambda^2\right)\chi(r, z, \overline{z}) = 0$. These new generators also satisfy the commutation relations of an $su(2)$ algebra, namely,

$$[\mathcal{S}_j, \mathcal{S}_k] = \mathrm{i}\epsilon_{jkl}\mathcal{S}_l \tag{28}$$

and the Casimir operators restricted to such subspaces with definite λ are given by

$$\mathcal{C}_1' = \partial_r^2 - \lambda^2\,\mathrm{e}^{-2r} - 2\lambda\mathcal{S}_2\,\mathrm{e}^{-r} + \mathcal{S}_3^2 - 1 \qquad \mathcal{C}_2' = \mathrm{i}\mathcal{S}_3\partial_r + \lambda\,\mathrm{e}^{-r}\mathcal{S}_1. \tag{29}$$

The symbol $\otimes$ has been used to stress the dual role played by these quantities. They are simultaneously matrices and differential operators. Hereafter, for simplicity, quantities such as $S_+ \otimes D$ will be written by juxtaposition, i.e. as S_+D.

The irreducible unitary representations of the group $SL(2, \mathbf{C})$ are characterized by the pair (j_0, j_1). For the principal series of unitary irreducible representations, $j_1 = \mathrm{i}k$ where $k \in \mathbf{R}$ and $j_0 = 0, \frac{1}{2}, 1, \ldots$ [16]. This set of representations describes scattering states of a charged particle moving on $\mathbf{H}$ in the field of the $SU(2)$ gauge field.

As a next step, recall that the eigenvalues of our Casimir operators are given by [16]

$$\mathcal{C}_1|j_0, j_1; \zeta\rangle = \left(j_0^2 + j_1^2 - 1\right)|j_0, j_1; \zeta\rangle \qquad \mathcal{C}_2|j_0 j_1; \zeta\rangle = -\mathrm{i}j_0, j_1|j_0, j_1; \zeta\rangle \tag{30}$$

where ζ represents additional labels to be specified later as required. For the particular realization to be used, it is useful to define the wavefunction as

$$\Psi_{j_0 j_1;\zeta}(r, z, \overline{z}) \equiv \langle r, z, \overline{z}|j_0 j_1; \zeta\rangle. \tag{31}$$

Note that $\Psi_{j_0 j_1;\zeta}(r, z, \overline{z})$ is a $2s+1$ component vector-valued wavefunction, its vector indices have been left implicit here. Using this wavefunction in equation (30) and restricting attention to the principal series of unitary irreducible representations, the eigenvalue equations for $\mathcal{C}_1'$ and $\mathcal{C}_2'$ yield

$$\left(-\partial_r^2 + \lambda^2\,\mathrm{e}^{-2r} + 2\lambda\,\mathrm{e}^{-r}\mathcal{S}_2\right)\Psi_{j_0 k;\zeta}(r, z, \overline{z}) = \left(k^2 + \mathcal{S}_3^2 - j_0^2\right)\Psi_{j_0 k;\zeta}(r, z, \overline{z}) \tag{32}$$

$$(\mathrm{i}\mathcal{S}_3\partial_r + \lambda\,\mathrm{e}^{-r}\mathcal{S}_1 - j_0 k)\Psi_{j_0 k;\zeta}(r, z, \overline{z}) = 0. \tag{33}$$

Formally identifying j_0^2 with the eigenvalue of $\mathcal{S}_3^2$ gives equation (32) as a Schrödinger-like equation for a one-dimensional scattering problem with a matrix-valued Morse-potential and at a scattering energy $E = k^2$.

4. Separation of variables

To perform the separation of variables note that the operator $-\triangle = P_1^2 + P_2^2$ is the Casimir operator of the Euclidean group $E(2)$ in two dimensions. This group is generated by the translation operators $P_1 = -\mathrm{i}\partial_x$, $P_2 = -\mathrm{i}\partial_y$ and the 'combined' rotation $\mathcal{G}_3 = -\mathrm{i}(x\partial_y - y\partial_x) + S_3$. Moreover, $\mathcal{G}_3$ is the generator of the $SO(2)$ subgroup of $E(2)$. Since $\triangle$ and $\mathcal{G}_3$ taken together with the Casimir operators $\mathcal{C}_1$ and $\mathcal{C}_2$ form a mutually commuting system of operators, further labels can be attached to the states forming $|j_0 j_1; \lambda m\rangle$. Those labels stem from the extra equations to be satisfied, namely,

$$(\triangle + \lambda^2)\Psi_{j_0 k;\lambda m}(r, z, \overline{z}) = 0 \qquad (\mathcal{G}_3 - m)\Psi_{j_0 k;\lambda m}(r, z, \overline{z}) = 0 \tag{34}$$

where m is integer or half-integer depending on the value of the isospin s. Consequently, the separation of variables can be achieved by using the subgroup chain $SO(2) \subset E(2) \subset SL(2, \mathbf{C})$.

The solution satisfying equations (34) is

$$\Phi^{\lambda s}_{mn}(\varrho, \theta) \equiv \chi^s_n \otimes Y^\lambda_{mn}(\varrho, \theta) \equiv \chi^s_n \otimes \mathrm{i}^{m-n} J_{m-n}(\lambda\varrho)\,\mathrm{e}^{\mathrm{i}(m-n)\theta} \qquad -s \leqslant n \leqslant s \tag{35}$$

as can be verified by introducing the polar coordinates $(x, y) = (\varrho\cos\theta, \varrho\sin\theta)$. Here a phase factor i^{m-n} has been introduced for convenience. For a fixed set of λ, s, m, n, $\Phi^{\lambda s}_{mn}(\varrho, \theta)$ is a $2s + 1$ component column vector with zeros everywhere except for the function $Y^\lambda_{mn}(\varrho, \theta)$ occupying the nth entry.

Moreover, in polar coordinates the operators in equation (27) take the form

$$D^\dagger = -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\left(\partial_w + \frac{\mathrm{i}}{w}\partial_\theta\right) \qquad D = -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\theta}\left(\partial_w - \frac{\mathrm{i}}{w}\partial_\theta\right) \tag{36}$$

where $w = \lambda\varrho$. Using the recursion relations in equation (3.1.27) on page 361 of [17]

$$\begin{aligned} \left(\frac{\mathrm{d}}{\mathrm{d}w} - \frac{(m-n)}{w}\right) J_{m-n}(w) &= -J_{m-(n-1)}(w) \\ \left(\frac{\mathrm{d}}{\mathrm{d}w} + \frac{(m-n)}{w}\right) J_{m-n}(w) &= J_{m-(n+1)}(w) \end{aligned} \tag{37}$$

gives

$$D^\dagger Y^\lambda_{mn}(\rho, \theta) = Y^\lambda_{mn-1}(\rho, \theta) \qquad DY^\lambda_{mn}(\rho, \theta) = Y^\lambda_{mn+1}(\rho, \theta). \tag{38}$$

$\mathcal{S}_\pm$ and $\mathcal{S}_3$ act as the usual $SU(2)$ generators on the functions $\Phi^{\lambda s}_{mn}(\varrho, \theta)$, since

$$\mathcal{S}_\pm\Phi^{\lambda s}_{mn}(\varrho, \theta) = \sqrt{(s \mp n)(s \pm n + 1)}\,\Phi^{\lambda s}_{mn\pm 1}(\varrho, \theta) \qquad \mathcal{S}_3\Phi^{\lambda s}_{mn}(\varrho, \theta) = n\Phi^{\lambda s}_{mn}(\varrho, \theta) \tag{39}$$

so that the group theoretical meaning of Φ can be clarified as follows.

The Euclidean group $E(2)$ is generated by a rotation with an angle φ and two translations by the vectors $(\varrho\cos\theta, \varrho\sin\theta)$. The unitary irreducible representations of $E(2)$ are classified by the purely imaginary number $\mathrm{i}\lambda$, $\lambda > 0$. According to equations (1) and (5) on page 168 of [18], the matrix elements of these unitary irreducible representations in the $SO(2)$ basis are

$$t^{\mathrm{i}\lambda}_{m'm}(\varphi, \varrho, \theta) = \mathrm{i}^{m-m'} J_{m-m'}(\lambda\varrho)\,\mathrm{e}^{-\mathrm{i}m\varphi}\,\mathrm{e}^{\mathrm{i}(m-m')\theta}. \tag{40}$$

Using equations (35) then, Φ can be expressed as

$$\Phi^{\lambda s}_{mn}(\varrho, \theta) = \chi^s_n \otimes t^{\mathrm{i}\lambda}_{nm}(0, \varrho, \theta) \qquad m \in \mathbf{Z} \quad \text{or} \quad m \in \tfrac{1}{2}\mathbf{Z} \quad -s \leqslant n \leqslant s. \tag{41}$$

Thus, an ansatz for the separation of variables can be made for the wavefunction $\Psi_{j_0 k,\lambda m s\alpha}(r, \varrho, \theta)$ $(-s \leqslant \alpha \leqslant s)$ satisfying equations (32) and (33). That ansatz is

$$\Psi_{j_0 k,\lambda m s\alpha}(r, \varrho, \theta) = \sum_{n=-s}^{s} \psi^{k j_0}_{\lambda s n}(r)\Phi^{\lambda s}_{mn\alpha}(\varrho, \theta) \qquad -s \leqslant \alpha \leqslant s. \tag{42}$$

By choosing the $SL(2, \mathbf{C})$ irreducible unitary representations as those labelled by the pair (k, j_0) where $k > 0$ and $-s \leqslant j_0 \leqslant s$ (they are inequivalent [16]), the radial part of the coupled channel wavefunction in the indices $-s \leqslant j_0, n \leqslant s$ becomes a $(2s + 1) \times (2s + 1)$ matrix ${\psi_n}^{j_0}(r)$. The remaining indices again have been suppressed for simplicity. The columns of ${\psi_n}^{j_0}(r)$ belong to inequivalent unireps of $SL(2, \mathbf{C})$ and specify different scattering boundary conditions. In mathematical language such objects are matroms of representations [19].

By virtue of equation (39), the matrix-valued differential operators $\mathcal{S}_\pm$ and $\mathcal{S}_3$ in the base as given by the $2s + 1$ base vectors Φ_n, take the usual matrix forms $S_\pm$ and S_3. Hence the radial versions of equations (32) and (33) can be written as

$$\left(\partial_r^2 + \lambda^2\, \mathrm{e}^{-2r} + 2\lambda\, \mathrm{e}^{-r} S_2 + k^2 + \mathrm{ad} S_3^2\right) \psi_{\lambda s}^k(r) = 0 \tag{43}$$

$$(\mathrm{i} S_3 \partial_r + \lambda\, \mathrm{e}^{-r} S_1) \psi_{\lambda s}^k(r) = k \psi_{\lambda s}^k(r) S_3 \tag{44}$$

where

$$\mathrm{ad} S_3^2 \psi_{\lambda n}^k(r) \equiv S_3^2 \psi_{\lambda s}^k(r) - \psi_{\lambda s}^k(r) S_3^2 \tag{45}$$

the matrix indices again being implicit. To solve equations (43) and (44), the group theoretical meanings of the separation ansatz (42) and of the coupled channel wavefunction $\psi_{\lambda s n}^{k j_0}(r)$ must be clarified.

As demonstrated in appendix A, the key equation in the construction of matrix-valued differential operators is the choice of section of the bundle $L(x, y, t)$. This is an $SL(2, \mathbf{C})$-valued function defined locally on the coset $\mathbf{H} \sim SL(2, \mathbf{C})/SU(2)$. Then as the Hamiltonian is related to the quadratic Casimir of $sl(2, \mathbf{C})$ with generators taken in the induced representation $\mathcal{U}$, finding the correct separation of variables ansatz for its eigenvalue problem amounts to doing harmonic analyses on $\mathbf{H}$. The theory of harmonic analyses on coset spaces [20] determines that the complete set of harmonics for a vector-valued field (carrying labels of the inducing representation of $SU(2)$) living on the coset $\mathbf{H}$ is given by matrix elements of the form $\langle j_0 k; s\nu | \mathcal{U}(L^{-1}(x, y, t)) | j_0 k; \lambda m\rangle$ where $-s \leqslant \nu \leqslant s$. For each fixed set of values of j_0, k, λ, m and s, this matrix element is a $2s + 1$ component quantity.

$L(x, y, t)$ is defined in the non-unitary finite-dimensional spinor representation of $SL(2, \mathbf{C})$ and can be written as

$$L(x, y, \mathrm{e}^{-r}) = \begin{pmatrix} 1 & x - \mathrm{i}y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-r/2} & 0 \\ 0 & \mathrm{e}^{r/2} \end{pmatrix}. \tag{46}$$

The generators of $SL(2, \mathbf{C})$ in this representation are $J_a = \frac{1}{2}\sigma_a$ and $K_\alpha = \frac{\mathrm{i}}{2}\sigma_\alpha$; details are given in appendix A. Introducing the two commuting 2×2 matrices

$$M_1 = K_1 - J_2 = \begin{pmatrix} 0 & \mathrm{i} \\ 0 & 0 \end{pmatrix} \qquad M_2 = K_2 + J_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{47}$$

permits $L(x, y, \mathrm{e}^{-r})$ to be recast in the form

$$L(x, y, \mathrm{e}^{-r}) = \mathrm{e}^{-\mathrm{i}(x M_1 + y M_2)}\, \mathrm{e}^{\mathrm{i} r K_3}. \tag{48}$$

As $L(x, y, \mathrm{e}^{-r})$ in the induced representation relates to $\mathcal{U}(L(x, y, \mathrm{e}^{-r}))$, (J_a, K_α) must be replaced by $(\mathcal{G}_a, \mathcal{F}_\alpha)$ and, as a consequence, (M_1, M_2) must be replaced by (P_1, P_2). In this context, note equations (3)–(8), (10) and (11). Hence

$$\mathcal{U}(L^{-1}(x, y, \mathrm{e}^{-r})) = \mathrm{e}^{-\mathrm{i} r F_3}\, \mathrm{e}^{\mathrm{i}(x P_1 + y P_2)}. \tag{49}$$

The matrix element $\langle j_0k; s\nu|\mathcal{U}(L^{-1}(x, y, \mathrm{e}^{-r})|j_0k; \lambda m\rangle$ then can be evaluated by inserting resolutions of the identity on C^{2s+1} (viewed as a subspace of the infinite-dimensional representation space), so that

$$\langle s\nu|\mathcal{U}(L^{-1}(x, y, \mathrm{e}^{-r}))|\lambda n\rangle = \sum_{n,n'=-s}^{s} \langle s\nu|sn'\rangle\langle sn'|\,\mathrm{e}^{-\mathrm{i}rF_3}|\lambda n\rangle\langle\lambda n|\,\mathrm{e}^{\mathrm{i}(xP_1+yP_2)}|\lambda m\rangle. \tag{50}$$

As F_3 acting on the base $|sn'\rangle$ shifts the values of s but leaves the value n' intact [16], $n = n'$ and the above is simply a sum over n. In that sum, the first and third terms are $\left(\chi^s_{n'}\right)_\nu$ and $t^{\mathrm{i}\lambda}_{nm}(0, \varrho, \theta)$, respectively, so that equation (50) recovers the separation ansatz (42), provided

$$\psi^{kj_0}_{\lambda sn}(r) = \mathrm{i}^{n+j_0}\,\mathrm{e}^r\langle j_0k; sn|\mathrm{e}^{-\mathrm{i}rF_3}|j_0k; \lambda n\rangle. \tag{51}$$

Note that this condition differs from that of equation (41) of [21] by a factor of $\mathrm{i}^{n+j_0}\,\mathrm{e}^r$. The second term in this product amounts to the similarity transformation introduced in equations (21), (22) and the first to inclusion of factors i^n into the representation matrix elements of P_1 (compare the relevant expressions in [21]).

However, unlike in the usual definition of the matrom [19], the matrix elements of the operator established herein are expressed in a mixed base, corresponding to the two different subgroup chains. The spin content is expressed by the chain $U(1) \subset SU(2) \subset SL(2, \mathbf{C})$ via the labels j_0, k, s, n. The other chain $SO(2) \subset E(2) \subset SL(2, \mathbf{C})$ with which labels j_0, k, λ, n are associated, represents the separation of variables. The important observation is that the matrix element, equation (51), is symmetric when regarded as a matrix in the indices j_0 and n. This can be easily proved using the results of [21]. Then

$$\psi^{kj_0}_{\lambda sn}(r) = \mathrm{i}^{n+j_0}\,\mathrm{e}^r N_{\lambda s}\frac{1}{2}\int_{-1}^{1} d(\cos\theta)d^s_{nj_0}(\theta)J_{n-j_0}\left(\lambda\,\mathrm{e}^{-r}\tan\frac{\theta}{2}\right)\left[\mathrm{e}^r\cos^2\frac{\theta}{2}\right]^{-\mathrm{i}k-1} \tag{52}$$

where $d^s_{nj_0}(\theta) = \langle sn|\,\mathrm{e}^{-\mathrm{i}\theta S_2}|sj_0\rangle$ and $N_{\lambda s}$ is a normalization factor independent of the values of (n, j_0), and as the relations

$$d^s_{nj_0}(\theta) = (-1)^{n-j_0}d^s_{j_0n}(\theta) \quad J_{n-j_0}(x) = (-1)^{j_0-n}J_{j_0-n}(x) \quad x = \lambda\,\mathrm{e}^{-r}\tan(\theta/2) \tag{53}$$

ensure that the integral is invariant with respect to a change of n and j_0,

$$\psi^{kj_0}_{\lambda sn}(r) = \psi^{kn}_{\lambda sj_0}(r). \tag{54}$$

Note that the form of the coupled channel wavefunction after integration [21] involves the Meier function G^{21}_{13} with arguments containing the index pair (j_0, n) and so carries the matrix structure of the coupled channel wavefunction in a complicated way. It was desirable to exhibit the channel structure more clearly in what follows, and so I sought another representation by continuing a study of the properties of the Casimir operators. Having two different representations for the coupled channel wavefunction is also useful when calculating the scattering matrix.

Using the symmetry relation (54) enables equation (43) to be replaced. By taking the transpose of equation (43) and using that with equation (54) yields $\left(\mathrm{ad}S_3^2\psi\right)^{\mathrm{T}} = -\left(\mathrm{ad}S_3^2\psi\right)$. Then by adding equation (43) to its transpose and using $S_2^{\mathrm{T}} = -S_2$, the alternate equation to equation (43) results, namely,

$$\left(\partial_r^2 - \lambda^2\,\mathrm{e}^{-2r} - \lambda\,\mathrm{e}^{-r}\mathrm{ad}S_2 + k^2\right)\psi^k_{\lambda s}(r) = 0. \tag{55}$$

The utility of this equation is that it contains only *one* of the matrices $\mathbf{S}$, i.e. S_2. Then with the variable

$$q = 2\lambda\,\mathrm{e}^{-r} \tag{56}$$

and the new function

$$\mathcal{R}^k_{\lambda s}(q) \equiv \sqrt{q}\,\psi^k_{\lambda s}(q) \tag{57}$$

equations (55) and (44) can be recast as

$$\left(q^2\partial_q^2 + \tfrac{1}{4} - \tfrac{1}{4}q^2 - \tfrac{1}{2}q\,(\mathrm{ad}S_2) + k^2\right)\mathcal{R}^k_{\lambda s}(q) = 0 \tag{58}$$

and

$$\left[\left(q\,\partial_q - \frac{1}{2}\right)S_3 + \frac{\mathrm{i}}{2}qS_1\right]\mathcal{R}^k_{\lambda s}(q) = \mathrm{i}k\mathcal{R}^k_{\lambda s}(q)S_3. \tag{59}$$

The first of these is a matrix analogue of Whittaker's equation [17],

$$\left(\partial_q^2 + \frac{1/4 + k^2}{q^2} - \frac{1}{4} + \frac{\kappa}{q}\right)F(q) = 0 \tag{60}$$

solutions of which are the Whittaker functions $W_{\kappa,\mathrm{i}k}(q)$ and $M_{\kappa,\pm\mathrm{i}k}(q)$ [17] and which satisfy $W_{\kappa,-\mathrm{i}k}(q) = W_{\kappa,\mathrm{i}k}(q)$. With the spectral projectors $\mathcal{P}_\alpha$ of S_2, as

$$S_2 = \sum_{\alpha=-s}^{s}\alpha\mathcal{P}_\alpha \qquad \mathcal{P}_\alpha\mathcal{P}_\beta = \delta_{\alpha\beta}\mathcal{P}_\beta \tag{61}$$

and as $S_2\mathcal{P}_\alpha = \alpha\mathcal{P}_\alpha$ and $\mathcal{P}_\alpha S_2 = \alpha\mathcal{P}_\alpha$, the ansatz

$$\mathcal{R}^k_{\lambda s\pm}(q) = \sum_{\alpha,\beta=-s}^{s}\mathcal{P}_\alpha C_\pm(k)\mathcal{P}_\beta M_{-\frac{1}{2}(\alpha-\beta),\pm\mathrm{i}k}(q) \qquad q = 2\lambda\,\mathrm{e}^{-r} \tag{62}$$

and a similar one containing the function $W_{-\frac{1}{2}(\alpha-\beta),\mathrm{i}k}(q)$ satisfy equation (58). Here $C_\pm(k)$ is a $(2s+1)\times(2s+1)$-matrix depending only on k. Equation (62) is called the general solution of equation (58). Note that $\mathcal{P}_\alpha$ can be represented as

$$\mathcal{P}_\alpha = \prod_{\gamma\neq\alpha,\gamma=-s}^{s}\frac{S_2-\gamma}{\alpha-\gamma} \tag{63}$$

yielding $\mathcal{P}_\alpha^{\mathrm{T}} = \mathcal{P}_{-\alpha}$. Hence equation (62) in this case assuredly is symmetric. But to also satisfy constraint (54), $C_\pm(k)$ must be symmetric. That will be shown, but, to gain insight into the nature of the matrix $C_\pm$, first it is instructive to work out the isospin-$\frac{1}{2}$ and 1 cases separately and by conventional means.

5. Special cases

5.1. The spin-$\frac{1}{2}$ case

As an alternative derivation for this special case consider the eigenvalue problem of $\mathcal{C}_2'$ that resulted in equation (59). For isospin $\frac{1}{2}$, $S_j = \frac{1}{2}\sigma_j$ and equation (59) becomes

$$\begin{pmatrix} q\,\partial_q - \frac{1}{2} & \frac{\mathrm{i}}{2}q \\ \frac{\mathrm{i}}{2}q & \frac{1}{2} - q\,\partial_q\end{pmatrix}\begin{pmatrix}\alpha(q) & \gamma(q) \\ \beta(q) & \delta(q)\end{pmatrix} = \mathrm{i}k\begin{pmatrix}\alpha(q) & -\gamma(q) \\ \beta(q) & -\delta(q)\end{pmatrix}. \tag{64}$$

The consistency of the resulting set of equations demands that $\alpha(q) = -\delta(q)$ and $\beta(q) = \gamma(q)$. Hence the pair of equations to be satisfied is

$$\left(q\,\partial_q - \frac{1}{2} - \mathrm{i}k\right)\alpha(q) + \frac{\mathrm{i}}{2}q\beta(q) = 0 \qquad \left(q\,\partial_q - \frac{1}{2} + \mathrm{i}k\right)\beta(q) - \frac{\mathrm{i}}{2}q\alpha(q) = 0. \tag{65}$$

As the recursion relations for the functions $M_{\kappa,\tau}(q)$ are [17]

$$\left(q\,\partial_q + \kappa - \tfrac{1}{2}q\right)M_{\kappa,\tau}(q) = \left(\tfrac{1}{2} + \tau + \kappa\right)M_{\kappa+1,\tau}(q) \tag{66}$$

$$\left(q\,\partial_q - \kappa + \tfrac{1}{2}q\right)M_{\kappa,\tau}(q) = \left(\tfrac{1}{2} + \tau - \kappa\right)M_{\kappa-1,\tau}(q) \tag{67}$$

with $\kappa = \pm\frac{1}{2}$ and $\tau = \pm\mathrm{i}k$, they are

$$\left(q\partial_q - \tfrac{1}{2} - \tfrac{1}{2}q\right) M_{-\frac{1}{2},\pm\mathrm{i}k}(q) = \pm\mathrm{i}kM_{+\frac{1}{2},\pm\mathrm{i}k}(q) \tag{68}$$

$$\left(q\partial_q - \tfrac{1}{2} + \tfrac{1}{2}q\right) M_{+\frac{1}{2},\pm\mathrm{i}k}(q) = \pm\mathrm{i}kM_{-\frac{1}{2},\pm\mathrm{i}k}(q). \tag{69}$$

Using similar relations for the functions $W_{\kappa,\tau}(q)$ [17] one finds

$$\begin{aligned}\left(q\partial_q - \tfrac{1}{2} + \tfrac{1}{2}q\right) W_{+\frac{1}{2},\mathrm{i}k}(q) &= k^2 W_{-\frac{1}{2},\mathrm{i}k}(q)\\ \left(q\partial_q - \tfrac{1}{2} - \tfrac{1}{2}q\right) W_{-\frac{1}{2},\mathrm{i}k}(q) &= -W_{+\frac{1}{2},\mathrm{i}k}(q).\end{aligned} \tag{70}$$

By adding and subtracting these relations, three different sets of solutions for equation (59) result. They are

$$\mathcal{R}_{\pm}(q) = \begin{pmatrix} M_{-\frac{1}{2},\pm\mathrm{i}k}(q) \pm M_{+\frac{1}{2},\pm\mathrm{i}k}(q) & \mathrm{i}\left(M_{-\frac{1}{2},\pm\mathrm{i}k}(q) \mp M_{+\frac{1}{2},\pm\mathrm{i}k}(q)\right) \\ \mathrm{i}\left(M_{-\frac{1}{2},\pm\mathrm{i}k}(q) \mp M_{+\frac{1}{2},\pm\mathrm{i}k}(q)\right) & -\left(M_{-\frac{1}{2},\pm\mathrm{i}k}(q) \pm M_{+\frac{1}{2},\pm\mathrm{i}k}(q)\right) \end{pmatrix} \tag{71}$$

and

$$\mathcal{R}(q) = \begin{pmatrix} W_{-\frac{1}{2},\mathrm{i}k}(q) + \frac{\mathrm{i}}{k}W_{+\frac{1}{2},\mathrm{i}k}(k) & \mathrm{i}\left(W_{-\frac{1}{2},\mathrm{i}k}(q) - \frac{\mathrm{i}}{k}W_{+\frac{1}{2},\mathrm{i}k}(q)\right) \\ \mathrm{i}\left(W_{-\frac{1}{2},\mathrm{i}k}(q) - \frac{\mathrm{i}}{k}W_{+\frac{1}{2},\mathrm{i}k}(q)\right) & -\left(W_{-\frac{1}{2},\mathrm{i}k}(q) + \frac{\mathrm{i}}{k}W_{+\frac{1}{2},\mathrm{i}k}(q)\right) \end{pmatrix}. \tag{72}$$

Using equation (57) the corresponding eigenfunctions of $\mathcal{C}_2'$ are also eigenfunctions to $\mathcal{C}_1'$.

To clarify the meaning of these solutions, consider the asymptotic behaviour of the coupled channel wavefunctions. First, note that the boundary of the upper half space, i.e. the set of points with $t = 0$ and $t = \infty$, is infinitely far away with respect to metric (2). Since $q = 2\lambda\,\mathrm{e}^{-r}$, taking the $|r| \to \infty$ limit of the one-dimensional scattering problem corresponds to the description of the wavefunction of the three-dimensional non-Abelian Landau problem on the boundary of $\mathbf{H}$. Recall that [17]

$$W_{\kappa,\mu}(q) = \frac{\Gamma(-2\mu)}{\Gamma(1/2-\mu-\kappa)}M_{\kappa,\mu}(q) + \frac{\Gamma(2\mu)}{\Gamma(1/2+\mu-\kappa)}M_{\kappa,-\mu}(q) \tag{73}$$

where

$$M_{\kappa,\mu}(q) = \mathrm{e}^{-\frac{1}{2}q}q^{\frac{1}{2}+\mu}{}_1F_1\left(\tfrac{1}{2}+\mu-\kappa; 1+2\mu; q\right) \tag{74}$$

in which ${}_1F_1(a;b;q) = 1 + \frac{a}{b}\frac{q}{1!} + \frac{a(a+1)}{b(b+1)}\frac{q^2}{2!} + \cdots$ is Kummer's function [17]. Using $q = 2\lambda\,\mathrm{e}^{-r}$, $\kappa = \pm\frac{1}{2}$ and $\mu = \mathrm{i}k$ gives

$$\mathcal{R}^k_{\lambda\frac{1}{2}}(r) = \frac{\Gamma(-2\mathrm{i}k)}{\Gamma(1-\mathrm{i}k)}\mathcal{R}^k_{\lambda\frac{1}{2};+}(r) + \frac{\Gamma(2\mathrm{i}k)}{\Gamma(1+\mathrm{i}k)}\mathcal{R}^k_{\lambda\frac{1}{2};-}(r). \tag{75}$$

Note that the limit $r \to -\infty$ yields vanishing wavefunctions due to the presence of the term $\mathrm{e}^{-\frac{1}{2}q}$ in equation (74). That is also clear since the interaction term is a combination of potentials of the forms e^{-r} and e^{-2r}, hence the interaction matrix goes to infinity as $r \to -\infty$. It follows that the only non-vanishing scattering quantities are the reflection coefficients for the two scattering channels. Reverting to the coupled channel wavefunction by using equation (57) and then taking the $r \to \infty$ limit gives

$$\lim_{r\to\infty}\psi^k_{\lambda\frac{1}{2}}(r) \sim (2\lambda)^{\mathrm{i}k}\frac{\Gamma(-2\mathrm{i}k)}{\Gamma(-\mathrm{i}k)}\,\mathrm{e}^{-\mathrm{i}kr} \otimes (-\sigma_3) + (2\lambda)^{-\mathrm{i}k}\frac{\Gamma(2\mathrm{i}k)}{\Gamma(\mathrm{i}k)}\,\mathrm{e}^{\mathrm{i}kr} \otimes (\mathrm{i}\sigma_1). \tag{76}$$

Clearly then $\psi_\pm(r)$ are the coupled channel analogues of the in- and out-going states, while $\psi(r)$ is the physical coupled channel wavefunction. Using the duplication formula for the Γ function, and normalizing the amplitude of the incoming plane wave part to unity gives us the scattering matrix

$$\lim_{r\to\infty} \psi^k_{\lambda\frac{1}{2}}(r) \sim \mathrm{e}^{-\mathrm{i}kr} \otimes I - \left(\frac{\lambda}{2}\right)^{-2\mathrm{i}k} \frac{\Gamma(1/2+\mathrm{i}k)}{\Gamma(1/2-\mathrm{i}k)} \mathrm{e}^{\mathrm{i}kr} \otimes \sigma_2 \tag{77}$$

where

$$\mathbf{S}_{\lambda\frac{1}{2}}(k) = -\left(\frac{\lambda}{2}\right)^{-2\mathrm{i}k} \frac{\Gamma(1/2+\mathrm{i}k)}{\Gamma(1/2-\mathrm{i}k)}\sigma_2 \tag{78}$$

a form which clearly identifies the scattering as a helicity scattering wherein a particle with isospin projection $\frac{1}{2}$ is scattered to an isospin projection $-\frac{1}{2}$, and vice versa.

This allows further insight into the structure of the general solution obtained in equation (62). Indeed, the expressions of equation (71) for the 'in' and 'out' solutions are in the form of equation (62) with the special choice

$$C_+ = -\sigma_3 \qquad C_- = \mathrm{i}\sigma_1 \tag{79}$$

which are now independent of k. Moreover, from equation (76), precisely the same $C_\pm$ matrices appear in the asymptotic form of the radial part of the coupled channel wavefunction.

Now consider the explicit form of the scattering solutions to the three-dimensional problem. For this the ϱ, θ dependence of the wavefunction must be considered. To within a normalization factor, from equations (40)–(42) those functions are

$$\Psi^{\frac{1}{2}k}_{\lambda m\frac{1}{2}}(r,\varrho,\theta) = \frac{\mathrm{i}^{m-1/2}}{\sqrt{2\lambda}} \mathrm{e}^{r/2} \begin{pmatrix} \left(\frac{\mathrm{i}}{k} W_{\frac{1}{2},\mathrm{i}k}(2\lambda\,\mathrm{e}^{-r}) + W_{-\frac{1}{2},\mathrm{i}k}(2\lambda\,\mathrm{e}^{-r})\right) J_{m-1/2}(\lambda\varrho)\,\mathrm{e}^{\mathrm{i}(m-1/2)\theta} \\ \left(\frac{\mathrm{i}}{k} W_{\frac{1}{2},\mathrm{i}k}(2\lambda\,\mathrm{e}^{-r}) - W_{-\frac{1}{2},\mathrm{i}k}(2\lambda\,\mathrm{e}^{-r})\right) J_{m+1/2}(\lambda\varrho)\,\mathrm{e}^{\mathrm{i}(m+1/2)\theta} \end{pmatrix} \tag{80}$$

and

$$\Psi^{-\frac{1}{2}k}_{\lambda m\frac{1}{2}}(r,\varrho,\theta) = \frac{-\mathrm{i}^{m+1/2}}{\sqrt{2\lambda}} \mathrm{e}^{r/2} \begin{pmatrix} \left(\frac{\mathrm{i}}{k} W_{\frac{1}{2},\mathrm{i}k}(2\lambda\,\mathrm{e}^{-r}) - W_{-\frac{1}{2},\mathrm{i}k}(2\lambda\,\mathrm{e}^{-r})\right) J_{m-1/2}(\lambda\varrho)\,\mathrm{e}^{\mathrm{i}(m-1/2)\theta} \\ \left(\frac{\mathrm{i}}{k} W_{\frac{1}{2},\mathrm{i}k}(2\lambda\,\mathrm{e}^{-r}) + W_{-\frac{1}{2},\mathrm{i}k}(2\lambda\,\mathrm{e}^{-r})\right) J_{m+1/2}(\lambda\varrho)\,\mathrm{e}^{\mathrm{i}(m+1/2)\theta} \end{pmatrix}. \tag{81}$$

Note that there are no bound states for the isospin $\frac{1}{2}$ case. That follows by diagonalizing the interaction term in equation (32), so obtaining the Morse potentials $\lambda^2\,\mathrm{e}^{-2r} + \lambda\,\mathrm{e}^{-r}$ and $\lambda^2\,\mathrm{e}^{-2r} - \lambda\,\mathrm{e}^{-r}$ $(\lambda > 0)$. Such can be done in the spin-$\frac{1}{2}$ case since, for that case, the $\mathrm{ad}S_3^2$ term is absent. Clearly the first of these potentials does not support the bound states. With regard to the second, recall that the unitary irreducible representations of $SL(2,\mathbf{C})$ fall into two distinct classes; the principal series which was required to describe the scattering states and the supplementary series which is characterized by the conditions $j_0 = 0$ and $0 \leqslant j_1 \leqslant 1$. Hence, the only possible choices omitted have $(j_0, j_1) = \left(0, \frac{1}{2}\right)$ and its equivalent mirror conjugate $\left(0, -\frac{1}{2}\right)$ describing zero modes, i.e. eigenstates with $E = 0$. The presence of such zero modes originates from the ones of equation (33). Reintroducing the variable $t = \mathrm{e}^{-r}$ and noting that $-\partial_r = t\partial_t$, readily shows that these eigenfunctions are of the form $\mathrm{e}^{\pm\lambda t}$. However, this eigenfunction is not square integrable for the relevant part of the hyperbolic measure being $\frac{1}{t^3}\,\mathrm{d}t$.

5.2. *The spin-1 case*

For the spin-1 case, consistency of equation (59) constrains the wavefunction to take the following form:

$$\mathcal{R}(q) = \begin{pmatrix} \alpha(q) & \beta(q) & \gamma(q) \\ \beta(q) & \delta(q) & -\beta(q) \\ \gamma(q) & -\beta(q) & \alpha(q) \end{pmatrix} \tag{82}$$

with the equations to be solved being

$$\left(q\partial_q - \frac{1}{2} - \mathrm{i}k\right)\alpha(q) + \frac{\mathrm{i}}{2}\frac{\sqrt{2}}{2}q\beta(q) = 0 \qquad \left(q\partial_q - \frac{1}{2} + \mathrm{i}k\right)\gamma(q) - \frac{\mathrm{i}}{2}\frac{\sqrt{2}}{2}q\beta(q) = 0 \tag{83}$$

and

$$\left(q\partial_q - \frac{1}{2}\right)\beta(q) + \frac{\mathrm{i}}{2}\frac{\sqrt{2}}{2}q\delta(q) = 0 \qquad \frac{\mathrm{i}}{2}\frac{\sqrt{2}}{2}q\left(\alpha(q) + \gamma(q)\right) = \mathrm{i}k\beta(q). \tag{84}$$

Recalling the relations

$$q\partial_q M_{0,\pm\mathrm{i}k}(q) = \tfrac{1}{2}\left(\tfrac{1}{2} \pm \mathrm{i}k\right)\left(M_{1,\pm\mathrm{i}k}(q) + M_{-1,\pm\mathrm{i}k}(q)\right) \tag{85}$$

derived from equations (68) and (69), and since [17]

$$qM_{0,\pm\mathrm{i}k}(q) = \left(\tfrac{1}{2} \pm \mathrm{i}k\right)\left(M_{-1,\pm\mathrm{i}k}(q) - M_{1,\pm\mathrm{i}k}(q)\right) \tag{86}$$

with the choice $\alpha(q) + \gamma(q) = M_{0,\pm\mathrm{i}k}(q)$, one finds

$$\beta_\pm(q) = \frac{1}{4\mathrm{i}k}\mathrm{i}\sqrt{2}qM_{0,\pm\mathrm{i}k}(q). \tag{87}$$

Adding equations (83) leads to $(q\partial_q - 1/2)(\alpha(q) + \gamma(q)) = \mathrm{i}k(\alpha(q) - \gamma(q))$, which, with $\alpha(q) + \gamma(q) = M_{0,\pm\mathrm{i}k}(q)$, in equation (85) gives

$$\alpha(q) - \gamma(q) = \frac{1}{2\mathrm{i}k}\left(-M_{0,\pm\mathrm{i}k}(q) + \left(\frac{1}{2} \pm \mathrm{i}k\right)\left(M_{1,\pm\mathrm{i}k}(q) + M_{-1,\pm\mathrm{i}k}(q)\right)\right). \tag{88}$$

Then from the first entries in equations (84) and (87), $\delta(q)$ is found. In all the results are

$$\alpha_\pm(q) = \frac{1}{4\mathrm{i}k}\left((-1 + 2\mathrm{i}k)M_{0,\pm\mathrm{i}k}(q) + \left(\frac{1}{2} \pm \mathrm{i}k\right)\left(M_{1,\pm\mathrm{i}k}(q) + M_{-1,\pm\mathrm{i}k}(q)\right)\right) \tag{89}$$

$$\gamma_\pm(q) = \frac{1}{4\mathrm{i}k}\left((1 + 2\mathrm{i}k)M_{0,\pm\mathrm{i}k}(q) - \left(\frac{1}{2} \pm \mathrm{i}k\right)\left(M_{1,\pm\mathrm{i}k}(q) + M_{-1,\pm\mathrm{i}k}(q)\right)\right) \tag{90}$$

and

$$\delta_\pm(q) = -\frac{1}{2\mathrm{i}k}\left(M_{0,\pm\mathrm{i}k}(q) + \left(\frac{1}{2} \pm \mathrm{i}k\right)\left(M_{1,\pm\mathrm{i}k}(q) + M_{-1,\pm\mathrm{i}k}(q)\right)\right). \tag{91}$$

A repeat of the argument given previously for the functions $W_{\kappa,\tau}(q)$ produces the similar set

$$\alpha(q) = \frac{1}{4\mathrm{i}k}\left((-1 + 2\mathrm{i}k)W_{0,\mathrm{i}k}(q) + \left(\frac{1}{2} + \mathrm{i}k\right)\left(\frac{1}{2} - \mathrm{i}k\right)W_{-1,\mathrm{i}k}(q) - W_{1,\mathrm{i}k}(q))\right) \tag{92}$$

$$\gamma(q) = \frac{1}{4\mathrm{i}k}\left((1 + 2\mathrm{i}k)W_{0,\mathrm{i}k}(q) - \left(\frac{1}{2} + \mathrm{i}k\right)\left(\frac{1}{2} - \mathrm{i}k\right)W_{-1,\mathrm{i}k}(q) + W_{1,\mathrm{i}k}(q))\right) \tag{93}$$

and

$$\delta(q) = -\frac{1}{2\mathrm{i}k}\left(W_{0,\mathrm{i}k}(q) + \left(\frac{1}{2} + \mathrm{i}k\right)\left(\frac{1}{2} - \mathrm{i}k\right)W_{-1,\mathrm{i}k}(q) - W_{1,\mathrm{i}k}(q))\right) \tag{94}$$

with

$$\beta(q) = \frac{1}{4\mathrm{i}k}\mathrm{i}\sqrt{2}q\,W_{0,\mathrm{i}k}(q) \tag{95}$$

that are to be used in the matrix equation (82). The full radial coupled channel wavefunction is then obtained by multiplying with $\frac{1}{\sqrt{q}}$.

To make contact with equation (62), the spectral projectors for the spin-1 case are required. Using the explicit form

$$S_2 = \mathrm{i}\frac{\sqrt{2}}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{96}$$

since $S_2^3 = S_2$, those projectors satisfy the usual relations

$$\mathcal{P}_\pm = \tfrac{1}{2}S_2(S_2 \pm I) \qquad \mathcal{P}_0 = (I - S_2)(I + S_2). \tag{97}$$

Also the choice

$$C_+(k) = \begin{pmatrix} \mathrm{i}k & 0 & 0 \\ 0 & -\mathrm{i}k - 1 & 0 \\ 0 & 0 & \mathrm{i}k \end{pmatrix} \qquad C_-(k) = \begin{pmatrix} 0 & 0 & \mathrm{i}k \\ 0 & \mathrm{i}k - 1 & 0 \\ \mathrm{i}k & 0 & 0 \end{pmatrix} \tag{98}$$

in equation (62) gives back the solutions of equation (87), and equations (89)–(91). That can be ascertained by using

$$C_+(k)S_2 = -S_2(C_+(k) + I) \qquad C_-(k)S_2 = -S_2(C_-(k) - E) \qquad E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{99}$$

to prove relations

$$\mathcal{P}_\pm C_+(k)\mathcal{P}_\mp = \mathcal{P}_\pm\left(C_+(k) + \tfrac{1}{2}I\right) \qquad \mathcal{P}_\pm C_+(k)\mathcal{P}_\pm = -\tfrac{1}{2}\mathcal{P}_\pm \qquad [\mathcal{P}_0, C_+(k)] = 0 \tag{100}$$

and

$$\mathcal{P}_\pm C_-(k)\mathcal{P}_\mp = \mathcal{P}_\pm\left(C_-(k) - \tfrac{1}{2}E\right) \qquad \mathcal{P}_\pm C_-(k)\mathcal{P}_\pm = \tfrac{1}{2}\mathcal{P}_\pm E \qquad [\mathcal{P}_0, C_-(k)] = 0 \tag{101}$$

with which equation (62) can be rewritten in terms of the explicit form of the projectors. Moreover, a relation similar to that of equation (75) holds, i.e.

$$\mathcal{R}^k_{\lambda 1}(r) = \frac{\Gamma(-2\mathrm{i}k)}{\Gamma(1/2 - \mathrm{i}k)}\mathcal{R}^k_{\lambda 1;+}(r) + \frac{\Gamma(2\mathrm{i}k)}{\Gamma(1/2 + \mathrm{i}k)}\mathcal{R}^k_{\lambda 1;-}(r) \tag{102}$$

with the channel structure of $\mathcal{R}$ and $\mathcal{R}_\pm$ as shown in equation (82). Since by using equation (74) it can be shown that

$$\lim_{r\to\infty}\alpha_-(r) = \lim_{r\to\infty}\gamma_+(r) = 0$$

the asymptotic form of the radial part of the coupled channel wavefunction is

$$\lim_{r\to\infty}\psi^k_{\lambda 1}(r) = (2\lambda)^{\mathrm{i}k}\frac{\Gamma(-2\mathrm{i}k)}{\Gamma(1/2 - \mathrm{i}k)}\,\mathrm{e}^{-\mathrm{i}kr} \otimes C_+(k) + (2\lambda)^{-\mathrm{i}k}\frac{\Gamma(2\mathrm{i}k)}{\Gamma(1/2 + \mathrm{i}k)}\,\mathrm{e}^{\mathrm{i}kr} \otimes C_-(k) \tag{103}$$

in which $C_{\pm}(k)$ is given in equation (98). The resulting scattering matrix is

$$\mathbf{S}_{\lambda 1}(k) = \left(\frac{\lambda}{2}\right)^{-2\mathrm{i}k} \frac{\Gamma(\mathrm{i}k)}{\Gamma(-\mathrm{i}k)} \begin{pmatrix} 0 & 0 & 1 \\ 0 & \frac{1-\mathrm{i}k}{1+\mathrm{i}k} & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{104}$$

Thus, both the isospin-$\frac{1}{2}$ and 1 cases yield scattering matrices that can be expressed as a product of two phase factors and a skew-diagonal matrix describing the helicity scattering process. This skew-diagonal matrix is of the form $C_{-}(k)C_{+}(k)^{-1}$. Hence to obtain a full description of the scattering process for an arbitrary isospin, a means to specify $C_{\pm}(k)$ needs to be found. Moreover, the matrices $C_{\pm}(k)$ are the only unknowns in equation (62).

6. The coupled channel wavefunction

While equation (62) defines the radial coupled channel wavefunction, from the foregoing the quantities unknown therein are the matrices $C_{\pm}(k)$. In view of the spin-$\frac{1}{2}$ and 1 results, it can be conjectured that these matrices in all cases should appear in the scattering matrix in the combination $C_{-}(k)C_{+}^{-1}(k)$ and with $C_{+}(k)$ being a *diagonal* and $C_{-}(k)$ a *skew-diagonal* matrix. That conjecture can be proved when the limit $r \to \infty$ of the radial equations (43) and (44) is taken, giving

$$\left(\partial_r^2 + k^2 + \mathrm{ad}S_3^2\right)\psi_{\lambda s}^{k\infty}(r) = 0 \qquad \mathrm{i}S_3\partial_r\psi_{\lambda s}^{k\infty}(r) = k\psi_{\lambda s}^{k\infty}(r)S_3. \tag{105}$$

Multiplying the second equation with S_3 from the *right*, and then its transpose from the *left*, using the symmetry property of ψ, and then subtracting the resulting two equations, gives

$$\left(\mathrm{ad}S_3^2\right)_{\alpha\gamma}\left(\psi_{\lambda s}^{k\infty}\right)_{\gamma\beta}(r) = \left[S_3^2, \psi_{\lambda s}^{k\infty}(r)\right]_{\alpha\beta} = 0 \qquad -s \leqslant \alpha, \beta, \gamma \leqslant s. \tag{106}$$

Putting this result into the first term of equation (106), results in the asymptotic form

$$\psi_{\lambda s}^{k\infty}(r) = A_{\lambda s;+}(k)\,\mathrm{e}^{\mathrm{i}kr} + A_{\lambda s;-}(k)\,\mathrm{e}^{-\mathrm{i}kr} \tag{107}$$

where the amplitudes $A_{\lambda s;\pm}(k)$ are the $(2s+1)\times(2s+1)$-matrices. Moreover, since S_3 is a diagonal matrix, equation (106) infers $(\alpha^2-\beta^2)\left(\psi_{\lambda s}^{k\infty}\right)_{\alpha\beta}(r) = 0$. Hence the only nonzero components of the coupled channel wavefunction $\psi_{\lambda s}^{k\infty}(r)$ have $\alpha = \pm\beta$. Using the ansatz of equation (107) in the second segment of equation (105) gives the amplitudes

$$S_3A_{\lambda s;\pm}(k) = \mp A_{\lambda s;\pm}(k)S_3 \qquad \text{i.e.} \quad (\alpha\pm\beta)(A_{\lambda s;\pm})_{\alpha\beta}(k) = 0. \tag{108}$$

These relations in conjunction with $(A_{\pm})_{\alpha\beta}$ only having non-zero components when $\alpha = \pm\beta$, yield the important result that $A_{-}(k)$ is a *diagonal* and $A_{+}(k)$ is a *skew-diagonal* matrix.

To relate to the matrices $C_{\pm}(k)$, note that because of equations (62) and (74), $\psi_{\lambda s\pm}^{k}(r) = \mathrm{e}^{r/2}\mathcal{R}_{\lambda s\pm}^{k}(r)$ asymptotes as

$$\lim_{r\to\infty}\psi_{\lambda s\pm}^{k}(r) = (2\lambda)^{\pm\mathrm{i}k}\,\mathrm{e}^{\mp\mathrm{i}kr}\sum_{\alpha,\beta=-s}^{s}\mathcal{P}_\alpha C_{\pm}(k)\mathcal{P}_\beta = (2\lambda)^{\pm\mathrm{i}k}\,\mathrm{e}^{\mp\mathrm{i}kr}C_{\pm}(k). \tag{109}$$

Also as $\sum_{\alpha=-s}^{s}\mathcal{P}_\alpha = I$, the channel structure of $C_{\pm}(k)$ is fixed by that of $A_{\mp}(k)$. Hence $C_{+}(k)$ is a diagonal, and $C_{-}(k)$ is a skew-diagonal matrix as expected.

To determine the unknown matrices $C_{\pm}(k)$, it is appropriate to use the two different representations given for the wavefunction. Form (62) when multiplied by $q^{-1/2}$ can be compared with the corresponding components of the representation in equation (52). Since the unknown quantities $C_{\pm}(k)$ in equation (62) characterize the asymptotic behaviour, these matrices can be extracted from the asymptotic limit of the equivalent representation (52). The normalization factor and other overall non-matrix-valued quantities are not relevant to this process. So the wavefunction of equation (52) must be re-expressed in a form suitable to take the limit $r \to \infty$. Details of this are given in appendix B. The result is

$$\lim_{r\to\infty} \psi^{kj_0}_{\lambda ns}(r) \sim \mathrm{i}^{2n} \left(\frac{\lambda}{2}\right)^{\mathrm{i}k} (-1)^{s+n} \left[\frac{\Gamma(n-\mathrm{i}k)}{\Gamma(n+1+\mathrm{i}k)} \frac{\Gamma(s+1+\mathrm{i}k)}{\Gamma(s+1-\mathrm{i}k)}\right] \delta_{n,j_0}\, \mathrm{e}^{-\mathrm{i}kr} + \left(\frac{\lambda}{2}\right)^{-\mathrm{i}k} (-1)^{s+n} \frac{\Gamma(n+\mathrm{i}k)}{\Gamma(n+1-\mathrm{i}k)} \delta_{n,-j_0}\, \mathrm{e}^{\mathrm{i}kr} \quad -s \leqslant n,\, j_0 \leqslant s. \tag{110}$$

To within an arbitrary normalization and phase factor of $-\mathrm{e}^{\mathrm{i}\pi s}$, this is

$$\lim_{r\to\infty} \psi^{kj_0}_{\lambda ns}(r) \sim \left(\frac{\lambda}{2}\right)^{\mathrm{i}k} \Gamma(\{s\}-\mathrm{i}k)(C_+)_{nj_0}(k)\, \mathrm{e}^{-\mathrm{i}kr} + \left(\frac{\lambda}{2}\right)^{-\mathrm{i}k} \Gamma(\{s\}+\mathrm{i}k)(C_-)_{nj_0}(k)\, \mathrm{e}^{\mathrm{i}kr} \tag{111}$$

where

$$(C_+)_{nj_0}(k) = -\mathrm{e}^{\mathrm{i}\pi s}(-1)^{s+n}\mathrm{i}^{n+j_0} \frac{\Gamma(n-\mathrm{i}k)}{\Gamma(\{s\}-\mathrm{i}k)} \frac{\Gamma(s+1+\mathrm{i}k)}{\Gamma(n+1+\mathrm{i}k)} \delta_{n,j_0} \tag{112}$$

and

$$(C_-)_{nj_0}(k) = -\mathrm{e}^{\mathrm{i}\pi s}(-1)^{s+n}\mathrm{i}^{n+j_0} \frac{\Gamma(n+\mathrm{i}k)}{\Gamma(\{s\}+\mathrm{i}k)} \frac{\Gamma(s+1-\mathrm{i}k)}{\Gamma(n+1-\mathrm{i}k)} \delta_{n,-j_0}. \tag{113}$$

Here $\{s\}$ denotes the fractional part of the isospin s. As a check, it is easy to see that equations (112) and (113) give back the results for the isospin-$\frac{1}{2}$ and 1 cases shown in equations (79) and (98).

Hence, I can conclude that the final form of coupled channel wavefunction is (to an arbitrary normalization)

$$\psi^k_{\lambda,s}(r) = \frac{1}{\sqrt{2\lambda}}\, \mathrm{e}^{r/2} \left(\Gamma(\{s\}+\mathrm{i}k)\mathcal{R}^k_{\lambda s;-}(r) + \Gamma(\{s\}-\mathrm{i}k)\mathcal{R}^k_{\lambda s;+}\right) \tag{114}$$

where $\mathcal{R}^k_{\lambda s;\pm}(r)$ is specified by equation (62), and $C_\pm(k)$ is to be taken from equations (112) and (113). Note that, and as shown in appendix C, the spectral projectors $\mathcal{P}_\alpha$ that are needed also in equation (62) can be represented as

$$(\mathcal{P}_\alpha)_{\mu\nu} = u_{\mu\alpha}\overline{u_{\nu\alpha}} \qquad u_{\mu\alpha} = \frac{\mathrm{i}^\mu 2^{-s}(2s)!}{\sqrt{(s+\mu)!(s-\mu)!(s+\alpha)!(s-\alpha)!}} K_{s+\alpha}\left(s+\mu;\, \frac{1}{2};\, 2s\right) \tag{115}$$

where u_α are the eigenvectors of the matrix S_2 belonging to the eigenvalue α and $K_{s+\alpha}(s+\mu;\, 1/2;\, 2s)$ are the Krawtchouk polynomials. An alternate expression can be derived by using other results [21] for $\psi^k_{\lambda s}(r)$. Such is given by equations (154) of appendix B, wherein also, for the isospin-$\frac{1}{2}$ case, a check is made that these representations of the wavefunction are identical. Note, however, that this equivalence of representations for arbitrary s could be highly nontrivial to develop, since one would like to establish relations between the Whittaker functions and the functions ${}_1F_2$.

However, the scattering matrix now can be determined easily from the asymptotic behaviour. With the details also given in appendix B, the result is

$$\mathcal{S}_{nn'}(\lambda, k, s) = \left(\frac{\lambda}{2}\right)^{-2\mathrm{i}k} \frac{\Gamma(n+\mathrm{i}k)}{\Gamma(n-\mathrm{i}k)} \frac{\Gamma(n+1+\mathrm{i}k)}{\Gamma(n+1-\mathrm{i}k)} \frac{\Gamma(s+1-\mathrm{i}k)}{\Gamma(s+1+\mathrm{i}k)} \mathrm{i}^{2n} \delta_{n,-n} \tag{116}$$

where $-s \leqslant n, n' \leqslant s$. Note that the λ and s dependence completely factorizes and that the process is a helicity scattering, i.e. a charged particle entering the channel with the isospin projection n is scattered to that with the isospin projection $-n$.

Note also that result (116) gives the scattering matrix for *both* of our models. However, we have to be a little bit careful. For our second model we have a *one-dimensional* scattering problem on a matrix-valued Morse potential. Since in this case we have no transmitted wave, equation (116) has to be interpreted as a matrix-valued reflection coefficent. The channel indices are to be interpreted as those due to some internal degrees of freedom. (For the one channel case, see, e.g. [1] in this respect.) For our first model we have a *three-dimensional* scattering problem. Moreover, in this case we have no potential, the scattering problem is purely geometrical. Hence, in order to develop a scattering theory and define a scattering matrix some alternative approach is needed. Luckily, the method of Lax and Phillips [25] can be applied precisely for such types of scattering problems. In particular, for the *two-dimensional* geometrical scattering problem on the upper half plane the theory has already been fully developed [26]. It has been shown that in this case our 'naive' definition of the scattering matrix is precisely the one that has to be used in the Lax–Phillips sense. Of course, our case is *three-dimensional* and also has some helicity degrees of freedom, so although a generalization of the Lax–Phillips method and the rigorous justification of our definition of the scattering matrix in principle can be done, but it is by no means a trivial task.

7. Motion on $\mathbf{H}/\Gamma$ in an $SU(2)$ gauge field

In this section, it is shown how an even larger class of exactly solvable models may be obtained by restricting the motion of a charged particle in the $SU(2)$ magnetic field equation (20) from $\mathbf{H}$ to a fundamental domain. That requires imposition of suitable boundary conditions on the coupled channel wavefunction. The basic idea is to consider *discrete* subgroups Γ of $SL(2,\mathbf{C})$ and to form the spaces $\mathbf{H}/\Gamma$ by identifying the points on the boundary of certain fundamental domains in $\mathbf{H}$. The arising manifolds are three-manifolds with constant negative sectional curvature. They are compact or non-compact (hence allowing scattering states) depending on the choice of Γ. Some of these three-manifolds have been used in the context of quantum chaos and cosmology [12, 13]. Those studies did not include a *non-Abelian* magnetic field. However there has been an attempt [27] to incorporate an *Abelian* $U(1)$ gauge field. Since the mathematics of the spectral theory of the operator C_1 on such manifolds is well known [14] and has been applied [12, 13], only the first steps towards an analogous theory for the operator $\mathcal{C}_1$ containing a non-Abelian gauge field will be outlined, though the correct boundary conditions for the coupled channel wavefunction will be specified.

Representing the points on $\mathbf{H}$ with the quaternion

$$w \equiv z + \mathrm{j}t = x + \mathrm{i}y + \mathrm{j}t \quad \text{where} \quad \mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1, \mathrm{ij} = -\mathrm{ji} = \mathrm{k} \tag{117}$$

and for an $PSL(2,\mathbf{C}) \sim SL(2,\mathbf{C})/\mathbf{Z}_2$ element, the fractional linear transformation representing the left action of $SL(2,\mathbf{C})$ on $\mathbf{H}$ can be written as

$$gw = (aw+b)(cw+d)^{-1} \qquad g \in \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{C}). \tag{118}$$

This transformation formula for the pair (z,t) is

$$g : (z,t) \mapsto \left(\frac{(az+b)(\overline{cz}+\overline{d}) + a\overline{c}t^2}{|cz+d|^2 + |c|^2t^2}, \frac{t}{|cz+d|^2 + |c|^2t^2} \right). \tag{119}$$

Let Γ be a discrete subgroup of $PSL(2,\mathbf{C})$. When gauge fields are not present, the *one-component* wavefunction must satisfy the boundary condition

$$\psi(gw) = \psi(w) \qquad g \in \Gamma \quad w \in \mathbf{H}. \tag{120}$$

Such functions are called automorphic functions with respect to Γ. When an $SU(2)$ gauge field is present with isospin of the gauge field being s, there are $2s + 1$ vector-valued wavefunctions, each having $2s + 1$ components. To find the analogue of equation (120), note that by construction our gauge field is the unique $SL(2, \mathbf{C})$ invariant $SU(2)$ gauge field [15]. Thus under an $SL(2, \mathbf{C})$ transformation of form (118), the one-form A remains invariant *up to a compensating $SU(2)$ gauge transformation*. The compensating local $SU(2)$ gauge transformation depends on the $SL(2, \mathbf{C})$ transformation in question. Forming three-manifolds $\mathbf{H}/\Gamma$ by gluing the boundaries of suitably chosen fundamental domains in $\mathbf{H}$, requires certain elements g of the discrete group Γ. Since non-Abelian gauge fields are to exist on these domains, the compensating gauge transformations corresponding to these gluing transformations of Γ need to be registered. A ramification is that there will be closed loops from which, on encircling, the multicomponent wavefunctions acquire an $SU(2)$ representation element. This $SU(2)$ matrix is precisely the compensating gauge transformation needed for the identification in the particular isospin s representation.

To find this compensating gauge transformation, the simplest (2×2)-matrix representation may be used. In this representation the one-form A and its gauge transform have the form

$$A = \frac{1}{2\mathrm{i}t}\begin{pmatrix} 0 & -\mathrm{d}\bar{z} \\ \mathrm{d}z & 0 \end{pmatrix} \qquad A' = U^\dagger A U - \mathrm{i}U^\dagger\, \mathrm{d}U \tag{121}$$

where A' is found by replacing the pair (z, t) in A with the $SL(2, \mathbf{C})$ transformed quantities (z', t') by using equation (119). $U(g; z, t)$ is the compensating gauge transformation to be found. It is a straightforward calculation to show that $U(g; z, t)$ is given by

$$U(g; z, t) = \frac{1}{\sqrt{|cz + d|^2 + |c|^2 t^2}}\begin{pmatrix} \overline{cz} + \overline{d} & -tc \\ t\bar{c} & cz + d \end{pmatrix}. \tag{122}$$

Then, to find the compensating gauge transformation for the general isospin s one can seek to represent $U(g; z, t)$ as

$$U(g; z, t) = \mathrm{e}^{\mathrm{i}\alpha_+ S_+}\, \mathrm{e}^{\mathrm{i}\alpha_3 S_3}\, \mathrm{e}^{\mathrm{i}\alpha_- S_-} \tag{123}$$

where $S_\pm = \frac{1}{2}\sigma_\pm$ and $S_3 = \frac{1}{2}\sigma_3$. In this way, the unknown quantities $\alpha_\pm$ and α_3 can be expressed in terms of the known ones $(a, b, c, d; z, t)$. A simple calculation shows that

$$\alpha_+ = \frac{\mathrm{i}tc}{cz + d} \qquad \alpha_- = \frac{-\mathrm{i}t\bar{c}}{cz + d} \qquad \mathrm{e}^{\frac{\mathrm{i}}{2}\alpha_3} = \frac{\sqrt{|cz + d|^2 + |c|^2 t^2}}{cz + d}. \tag{124}$$

Using defined relations [28]

$$|s\mu\rangle = \sqrt{\frac{(s-\mu)!(s+\mu)!}{(2s)!}}\frac{(S_+)^{s+\mu}}{(s+\mu)!}|s - s\rangle = \sqrt{\frac{(s-\mu)!(s+\mu)!}{(2s)!}}\frac{(S_-)^{s-\mu}}{(s-\mu)!}|ss\rangle \tag{125}$$

the general explicit formula for the compensating gauge transformation is

$$\mathcal{U}_{\mu\nu}(g; w) = \sqrt{\frac{(s+\mu)!(s+\nu)!}{(s-\mu)!(s-\nu)!}}(-\alpha_+)^\mu(\alpha_-)^\nu \sum_{\varrho=-s}^{s} \frac{(s-\varrho)!}{(s+\varrho)!(\mu-\varrho)!(\nu-\varrho)!}(1 + |\alpha_+|^2)^\varrho \tag{126}$$

where the parameters $\alpha_\pm$ depend on w and the group parameters are given by equation (124).

The boundary condition needed on three-manifolds of the form $\mathbf{H}/\Gamma$ when an $SU(2)$ gauge field of the form (121) is present, is

$$\psi_\mu(gw) = \mathcal{U}_{\mu\nu}(g; w)\psi_\nu(w) \quad w \equiv (z, t) \in \mathbf{H} \quad g \in \Gamma \subset PSL(2, \mathbf{C}) \quad -s \leqslant \mu, \nu \leqslant s. \tag{127}$$

Here $\mathcal{U}_{\mu\nu}(g; w)$ is given by equation (126). Note that on the boundary of $\mathbf{H}$ characterized by the restriction $t = 0$ (it is just the one-point compactification of $\mathbf{R}^2$, i.e. S^2, the two sphere) this boundary condition has the diagonal form

$$\psi_\mu(gw) = \mathrm{e}^{-2\mathrm{i}\mu\arg(cz+d)}\psi_\mu(w) \quad -s \leqslant \mu \leqslant s. \tag{128}$$

Hence the transformation formula (127) can be regarded as the extension of that used previously [9] for the Abelian $U(1)$ gauge field describing the constant magnetic field on the upper half plane. The transform extends from $\mathbf{R}^2$ to $\mathbf{H}$. However, this extension unlike that of equation (128) mixes the different isospin components.

The nature and mathematical meaning of the boundary condition (127) is revealed on recalling equation (122) for the isospin-$\frac{1}{2}$ case, and noting that for the isospin-1 case

$$\mathcal{U}(g; z, t) = \frac{1}{|cz+d|^2 + |c|^2t^2}\begin{pmatrix} (\overline{cz}+\overline{d})^2 & -\sqrt{2}ct(\overline{cz}+\overline{d}) & c^2t^2 \\ \sqrt{2}\overline{c}t(\overline{cz}+\overline{d}) & |cz+d|^2 - |c|^2t^2 & -\sqrt{2}ct(cz+d) \\ \overline{c}^2t^2 & \sqrt{2}\overline{c}t(cz+d) & (cz+d)^2 \end{pmatrix}. \tag{129}$$

Introducing a new wavefunction,

$$\phi_\mu(z, t) \equiv t^{-2s}\psi_\mu(z, t) \qquad \text{where} \quad s = \tfrac{1}{2}, 1 \tag{130}$$

facilitates this study, as then for isospin $\frac{1}{2}$,

$$\begin{aligned} \phi_{1/2}(gw) &= (\overline{cz}+\overline{d})\phi_{1/2}(w) - ct\phi_{-1/2}(w) \\ \phi_{-1/2}(gw) &= \overline{c}t\phi_{1/2}(w) + (cz+d)\phi_{-1/2}(w). \end{aligned} \tag{131}$$

Similarly, there is a simple form for the isospin-1 case. That contains only real-valued polynomial coefficients of second order in ct, in $cz+d$ and in their conjugates; a pattern that survives for general s since the 'trick' of equation (130) always gives new wavefunctions with a transformation formula containing $2s$ order polynomials in ct, in $cz+d$ and in their conjugates. This structure is reminiscent of a well-known formula in the theory of automorphic forms of degree $2s$ on the upper half plane $\mathbf{U}$ [29], i.e.

$$\varphi(gz) = (cz+d)^{2s}\varphi(z) \qquad z \in \mathbf{U} \quad g \in \Gamma \subset PSL(2, \mathbf{R}). \tag{132}$$

This encapsulates the correct boundary conditions on two-manifolds, i.e. Riemann-surfaces $\mathbf{U}/\Gamma$, when a constant magnetic field with quantized values $B \equiv 2s$ exists. Exploiting quaternions for $s = \frac{1}{2}$, equation (131) can also be cast in this form. Indeed, reintroducing the quaternion $w = x + \mathrm{i}y + \mathrm{j}t$, and defining two-by-two matrices by

$$\begin{aligned} &I \equiv -\mathrm{i}\sigma_3 \quad J \equiv -\mathrm{i}\sigma_2 \quad W = x\mathbf{1} + yI + tJ \\ &C \equiv (\mathrm{Re}\, c)\mathbf{1} + (\mathrm{Im}\, c)I \quad D \equiv (\mathrm{Re}\, d)\mathbf{1} + (\mathrm{Im}\, d)I \end{aligned} \tag{133}$$

where $\mathbf{1}$ is the two-by-two identity matrix, equation (131) takes the form resembling the quaternion analogue of (132) for $s = \frac{1}{2}$,

$$\phi_\mu(gw) = (CW + D)_{\mu\nu}\phi_\nu(w) \qquad w \in \mathbf{H} \quad g \in \gamma \subset PSL(2, \mathbf{C}). \tag{134}$$

Note also that the matrix $\mathcal{U}(g, w)$ for arbitrary s satisfies the consistency (co-cycle) condition

$$\mathcal{U}(g_1g_1, w) = \mathcal{U}(g_2, w)\mathcal{U}(g_1, g_2w) \qquad g_1, g_2 \in \Gamma \quad w \in \mathbf{H} \tag{135}$$

a condition which is also satisfied by the automorphy factor appearing in equation (132) in analogy with the usual theory of modular forms [29].

In all, for different possible choices of Γ, a large class of exactly solvable models defined on three-manifolds of the type $\mathbf{H}/\Gamma$ have been found. These models describe the propagation of a charged particle on these manifolds under the influence of a non-Abelian $SU(2)$ gauge field. Solutions of these models equate to finding solutions to the eigenvalue problems of Casimir operators as given by equation (16), and which satisfy the boundary conditions of equation (127). When *no* magnetic field is present in these models, it is well known [30] that, for a suitably chosen class of Γ allowing scattering states, these solutions are given explicitly by the Eisenstein series. In such cases, the asymptotic limit of the wavefunction and the scattering matrix can be defined and calculated. However, fine details of the scattering matrix depend on the number of theoretic properties of Γ [30]. When a non-Abelian magnetic-field is present, however, these results need to be, and can be, generalized by using a suitable generalization of the Eisenstein series. Herein, my intention was to derive the correct form of the boundary condition. Further details of such solvable models will be given in a future publication.

8. Conclusions and comments

Two exactly solvable models of multichannel scattering have been considered. The first describes the non-relativistic scattering of a charged particle on the curved manifold $\mathbf{H}$, the Poincaré upper half space with constant negative sectional curvature under the influence of an $SU(2)$ gauge field. The second model considers one-dimensional potential scattering with a matrix-valued interaction term that contains Morse-like potentials. These models have an equivalence that is based on a coset space construction and on the fact that different sets of local coordinates can be used for the coset. The associated coupled channel wavefunctions have been constructed in two different ways. The first was based upon an explicit construction regarding these wavefunctions as simultaneous eigenfunctions of the $SL(2, \mathbf{C})$ Casimir operators. The second method of construction originates from the observation that $\mathbf{H}$ is just the coset $SL(2, \mathbf{C})/SU(2)$ so that known results from the theory of harmonic analysis on coset spaces can be used. Using these representations, the explicit form of the scattering matrix valid for an arbitrary isospin s has been ascertained.

The exact solutions presented herein may have interesting applications in cosmology, since the change of coordinates

$$X_1 = \frac{x}{t} \quad X_2 = \frac{y}{t} \quad X_3 = \frac{x^2 + y^2 + t^2 - 1}{2t} \quad X_4 = \frac{x^2 + y^2 + t^2 + 1}{2t} \tag{136}$$

establishes a homeomorphism between the upper half space $\mathbf{H}$ and the upper sheet of the double-sheeted hyperboloid defined by $(X_1)^2 + (X_2)^2 + (X_3)^2 - (X_4)^2 = -1$ and $X_4 \geqslant 1$. Expressing the vector $(\mathbf{X}, X_4)$ in terms of polar coordinates as $\mathbf{X} = \sinh\rho\mathbf{n}$ and $X_4 = \cosh\rho$, with $\mathbf{n} \equiv (n_1, n_2, n_3) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ metric (2) becomes

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \sinh^2\rho(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{137}$$

Then as the cosmological line elements comply with the principles of isotropy and homogeneity (Robertson–Walker geometries) it can be written in the form

$$\mathrm{d}\sigma^2 = -c^2\,\mathrm{d}\tau^2 + R(\tau)\,\mathrm{d}s^2 \tag{138}$$

where $\mathrm{d}s^2$ is the line element of a three-dimensional space of constant curvature, assumed to be negative. Also $R(\tau)$ is the expansion factor which controls the Gaussian curvature of the space-like slices $\tau = \text{const}$. (Here $x^0 = \tau$ corresponds to the time variable and is not to be confused with $x^3 = t$.) So, while for such models $\mathrm{d}s^2$ is usually written in the form of equation (137), these space-like slices can be modelled by $\mathbf{H}$ as well using metric (2).

Likewise, the problem of Maxwell's equations on a cosmological background of the form $\mathbf{R}^{(+)} \times \mathbf{H}/\Gamma$ has been considered before [27]. For this, the space-like slices are topologically nontrivial three-manifolds, and to obtain solutions of the corresponding $U(1)$ gauge theory on such spaces the gauge-potentials need to be periodized with respect to the gluing transformations of Γ. This was done using the vector-field property of A_μ. However, besides being a vector A_μ is also a gauge field, hence a periodization *up to an* $U(1)$ *gauge transformation* would have been more adequate. In this respect the non-Abelian gauge field described herein behaves naturally. Indeed, the gauge field (17) is the *unique* $SL(2,\mathbf{C})$ invariant $SU(2)$-valued gauge field on $\mathbf{H}$ [31]. Moreover, the corresponding two-form F also satisfies the Yang–Mills equations on $\mathbf{H}$. Since an invariant gauge field is invariant up to a compensating $SU(2)$ gauge transformation, these objects are just the natural ones on spaces of the form $\mathbf{H}/\Gamma$, with $\Gamma \subset SL(2,\mathbf{C})$. Hence in these cosmological models instead of using an $U(1)$ gauge field, it seems more natural to consider an $SU(2)$ one.

As far as the problem of finding possible solutions of the $SU(2)$ gauge theory on $\mathbf{R}^{(+)} \times \mathbf{H}/\Gamma$ spaces is concerned, note that the static choice $A_j = (A_0, \mathbf{A}(\mathbf{x})) = (0, \mathbf{A}(\mathbf{x}))$ with $j = 0, 1, 2, 3$, also is a solution of the Yang–Mills equations on the $\mathbf{R}^{(+)} \times \mathbf{H}$ Robertson–Walker spaces. Since $\mathcal{A} \equiv A_j\,\mathrm{d}x^j$ is still an $SL(2,\mathbf{C})$ invariant gauge field (the time-independent compensating gauge transformations leave the $A_0 = 0$ gauge invariant), this development also gives a solution on $\mathbf{R}^{(+)} \times \mathbf{H}/\Gamma$.

Solutions of the Klein–Gordon wave equation in the static field $\mathcal{A}$ can also be considered in the spirit of [13], i.e.

$$\left(-\frac{c^2}{R^3(\tau)}\left(\frac{\partial}{\partial\tau}R^3(\tau)\frac{\partial}{\partial\tau}\right) + \frac{1}{R^2(\tau)}\triangle(A) - \left(\frac{mc}{\hbar}\right)^2 - \xi\hat{R}\right)\Psi(\tau, x, y, t) = 0 \qquad (139)$$

where $\triangle(A)$ is just the Laplace–Beltrami operator in the gauge field (17), $\hat{R}$ is the curvature scalar of the metric (138) and ξ is a dimensionless parameter that couples Ψ to the curvature scalar. Separation of variables in the form $\Psi(\tau, \mathbf{x}) = \chi(\tau)\psi(\mathbf{x})$ with $\mathbf{x} \equiv (x, y, t)$ gives the equation $(\triangle(A)+\lambda)\psi(\mathbf{x}) = 0$, with λ being the separation parameter. The spectral problem of this equation on $\mathbf{H}$ is precisely that solved herein. The corresponding equation for $\chi(\tau)$ can be found in [13]. It would be interesting to attempt to solve this equation on $\mathbf{R}^{(+)} \times \mathbf{H}/\Gamma$ as well. To do so, solutions $\psi(\mathbf{x})$ satisfying the more general boundary conditions (127) must be found. Note that these are the generalizations of the Γ periodic boundary condition used before [13] and from which λ_0, the eigenvalue corresponding to the only square integrable bound state, was found to be connected with the Hausdorff dimension of the limit set of the group Γ. This limit set forms a quasi-self-similar curve on the boundary of $\mathbf{H}$ and the class of trajectories with end points lying in this limit set is chaotic. In this picture this fractal-structured limit set provides the link between the classical chaos and the existence of a localized quantum state for the Klein–Gordon field. It is important to investigate how these results (e.g. the number of localized states as a function of the isospin s) are modified when a non-Abelian gauge field is also present.

In closing, note that the gauge field of equation (17) is just the pull-back of the so-called H-connection with respect to the section, equation (146), a connection that can be defined on any principal bundle G over the coset G/H. Moreover, on G/H there is a natural Riemannian metric coming from the Cartan–Killing metric of the semi-simple Lie algebra $\mathbf{g}$ of G. For the special case at hand with $G/H \simeq SL(2,\mathbf{C})/SU(2)$ this metric is just that given by equation (2). On a Riemannian manifold with a metric, the (so-called) spin connection also can be defined. This connection is needed for a consistent implementation of spinors on a curved space, which enables a correct definition of the Dirac operator D on the manifold. Since the gamma matrices in three dimensions are just the two-by-two Pauli matrices, some connection between the Dirac operator for a mass zero fermion on the curved space $\mathbf{H}$ and the *first-order differential operator* $\mathcal{C}_2$ of equation (22) for the isospin-$\frac{1}{2}$ case can be expected. Indeed, a straightforward calculation shows that the two connections for $\mathbf{H}$, the H-connection for isospin $\frac{1}{2}$ and the spin connection are identical. Hence $2\mathcal{C}_2 = D$ which implies that the eigenfunctions of the *relativistic* scattering problem of a *free* massless fermion have the same form, equations (80), (81), as the eigenfunctions of the *non-relativistic* problem of a spinless particle *in a non-Abelian gauge field with isospin* $\frac{1}{2}$. This identification of the spin and H-connections is not a coincidence. According to a theorem [20], the two connections coincide if G/H is a *symmetric space*, a condition which holds in our case.

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Appendix A

The construction of a matrix-valued differential realization of the semi-simple Lie-algebra $\mathbf{g}$ is based on a particular choice of local section of the principal bundle G over G/H with fibre H. This is a mapping $L : \mathcal{U} \to G, x \mapsto L(x)$ where $x \in \mathcal{U}$ is a local coordinate in an open neighbourhood of G/H. The choice of this section is not unique. For an arbitrary function $h : \mathcal{U} \to H$, the new section $L'(x) \equiv L(x)h(x)$ gives rise to a different realization of $\mathbf{g}$. However, since the generators constructed are gauge covariant, the new realization based on L' is gauge equivalent to that constructed from L.

Decompose the Lie-algebra as

$$\mathbf{g} = \mathbf{h} \oplus \mathbf{m} \tag{140}$$

where $\mathbf{h}$ is the sub-algebra corresponding to H and $\mathbf{m}$ is the orthogonal complement of $\mathbf{h}$ with respect to the Cartan–Killing metric. Then on splitting the generators of $\mathbf{g}$ into two corresponding subsets $J_a, a = 1, 2, \ldots, \dim \mathbf{h}$ and $K_\alpha, \alpha = 1, 2, \ldots, \dim \mathbf{m}$, commutation relations in this base are

$$[J_a, J_b] = \mathrm{i}C^c_{ab} J_c \quad [J_a, K_\alpha] = \mathrm{i}C^\beta_{a\alpha} K_\beta \quad [K_\alpha, K_\beta] = \mathrm{i}C^a_{\alpha\beta} J_a \tag{141}$$

where it is assumed that G/H is a *symmetric space*. So no K terms appear in the third set of commutation relations, i.e. $C^\gamma_{\alpha\beta} = 0$. Note that the construction is valid for the non-symmetric space case as well.

Next, for a particular choice of $L(x)$, define the quantities $\mathcal{D}^J_I(x), I, J = 1, 2 \ldots, \dim \mathbf{g}$, $A^a_\mu(x) J_a \, \mathrm{d}x^\mu$, $E^\alpha_\mu(x) K_\alpha \, \mathrm{d}x^\mu$, $\mu = 1, 2, \ldots, \dim \mathbf{m} = \dim G/H$ for which

$$L^{-1}(x) J_a L(x) = \mathcal{D}^b_a(x) J_b + \mathcal{D}^\alpha_a(x) K_\alpha \quad L^{-1}(x) K_\alpha L(x) = \mathcal{D}^b_\alpha(x) J_b + \mathcal{D}^\beta_\alpha(x) K_\beta \tag{142}$$

and

$$\mathrm{i}L^{-1}(x) \, \mathrm{d}L(x) = A^a_\mu(x) J_a \, \mathrm{d}x^\mu + E^\alpha_\mu(x) K_\alpha \, \mathrm{d}x^\mu. \tag{143}$$

On choosing an irreducible unitary matrix representation D for H, the generators of the induced representation of G induced by D can be constructed [15]. They are matrix-valued differential operators (covariant Lie-derivatives) of the form

$$G_a = -\mathrm{i}\mathcal{D}^\alpha_a(x) E^\mu_\alpha(x) \left(\partial_\mu - \mathrm{i}A^b_\mu D(J_b)\right) + \mathcal{D}^b_a(x) D(J_b) \tag{144}$$

and

$$F_\alpha = -\mathrm{i}\mathcal{D}^\beta_\alpha(x)E^\mu_\beta(x)\left(\partial_\mu - \mathrm{i}A^b_\mu D(J_b)\right) + \mathcal{D}^b_\alpha(x)D(J_b) \tag{145}$$

where $D(J_a)$ are the generators of the sub-algebra $\mathbf{h}$ in the inducing representation D and $E^\mu_\alpha(x)$ is the inverse matrix of $E^\alpha_\mu(x)$. It can be shown that the generators G_a and F_α satisfy the commutation relations (141), hence giving a realization of the Lie-algebra $\mathbf{g}$. For this case, $G = SL(2,\mathbf{C})$, $H = SU(2)$ and $G/H \simeq \mathbf{H}$. The ranges of indices are $I, J = 1,\ldots,6$, $a, b = 1, 2, 3$, $\alpha, \beta, \mu = 1, 2, 3$. The inducing representation is the usual $(2s+1)$-dimensional one with which $D(J_a) \equiv S_a$. Choosing the section as

$$L(x,y,t) = \begin{pmatrix} \sqrt{t} & \frac{x-\mathrm{i}y}{\sqrt{t}} \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix} \in SL(2,\mathbf{C}) \tag{146}$$

the $SL(2,\mathbf{C})$ generators in the (non-unitary) 2×2 defining representation as $J_a = \frac{1}{2}\sigma_a$, and $K_\alpha = \frac{\mathrm{i}}{2}\sigma_\alpha$, a straightforward calculation shows that

$$A^a_\mu(x)J_a\,\mathrm{d}x^\mu = \frac{1}{t}(J_1\,\mathrm{d}y - J_2\,\mathrm{d}x) \quad E^\alpha_\mu(x) = \frac{1}{t}\delta^\alpha_\mu \quad E^\mu_\alpha(x) = t\delta^\mu_\alpha. \tag{147}$$

Then the quantities $\mathcal{D}^J_I(x)$ can be expressed as

$$\mathcal{D}^b_a(x) = \mathcal{D}^\beta_\alpha(x) = \begin{pmatrix} \frac{1-x^2+y^2+t^2}{2t} & -\frac{xy}{t} & -x \\ -\frac{xy}{t} & \frac{1+x^2-y^2+t^2}{2t} & -y \\ \frac{x}{t} & \frac{y}{t} & 1 \end{pmatrix} \tag{148}$$

and

$$\mathcal{D}^a_\alpha(x) = -\mathcal{D}^\alpha_a(x) = \begin{pmatrix} \frac{xy}{t} & \frac{1-x^2+y^2-t^2}{2t} & y \\ -\frac{1+x^2-y^2-t^2}{2t} & -\frac{xy}{t} & -x \\ -\frac{y}{t} & \frac{x}{t} & 0 \end{pmatrix}. \tag{149}$$

Collecting everything together and substituting into equations (144) and (145) determines the realization, equations (10) and (11), used in section 2.

Appendix B

Consider the wavefunction given in the paper of Delbourgo *et al* [21], i.e.

$$\psi^{kj_0}_{\lambda sn}(r) = \mathrm{i}^{j_0+n}\,\mathrm{e}^r\,\frac{N^k_{\lambda s}}{2}\int_{-1}^{1}\mathrm{d}(\cos\theta)d^s_{nj_0}(\theta)J_{n-j_0}\left(\lambda\,\mathrm{e}^{-r}\tan\frac{\theta}{2}\right)\left[\mathrm{e}^r\cos^2\frac{\theta}{2}\right]^{-\mathrm{i}k-1} \tag{150}$$

where

$$N^k_{\lambda s} = \sqrt{-(2s+1)\lambda\Gamma(-s-\mathrm{i}k)\Gamma(s+1-\mathrm{i}k)\cos\pi(\mathrm{i}k+\lambda/2)}\left(\frac{\lambda}{2}\right)^{\mathrm{i}k}. \tag{151}$$

Changing the variables to $u = \cos\theta$ and using [22]

$$d^s_{nj_0}(u) = \sum_j(-1)^{j+n-j_0}\Delta_{sj,nj_0}[(1+u)/2]^{s-j+(j_0-n)/2}[(1-u)/2]^{j-(j_0-n)/2} \tag{152}$$

where

$$\Delta_{sj,nj_0} = \frac{\sqrt{(s+n)!(s-n)!(s+j_0)!(s-j_0)!}}{(s-n-j)!(s+j_0-j)!(j+n-j_0)!j!} \tag{153}$$

and with the new variable $w = \frac{1-u}{1+u}$, this wavefunction becomes

$$\psi^{kj_0}_{\lambda sn}(t) = \mathrm{i}^{j_0+n} t^{\mathrm{i}k} \sum_{j=0}^{\infty} (-1)^{j+n-j_0} \Delta_{sj,nj_0} \int_0^{\infty} \mathrm{d}w \frac{w^{j+(n-j_0)/2}}{(1+w)^{s+1-\mathrm{i}k}} J_{n-j_0}(\lambda t\sqrt{w}). \tag{154}$$

As usual, the sum over j is limited to values for which all arguments of factorials are not negative.

Next, after the substitution $w = z^2$ the result [23], page 710,

$$\begin{aligned} 2\int_0^{\infty} \mathrm{d}y \frac{y^{\varrho-1}}{(1+y^2)^{\mu+1}} J_\nu(ay) &= \left(\frac{a}{2}\right)^{\nu} \frac{\Gamma((\varrho+\nu)/2)\Gamma(\mu+1-(\varrho+\nu)/2)}{\Gamma(\mu+1)\Gamma(\nu+1)} {}_1F_2 \\ &\times \left(\frac{(\varrho+\nu)}{2}; \frac{(\varrho+\nu)}{2} - \mu, \nu+1; \frac{a^2}{4}\right) + \left(\frac{a}{2}\right)^{2\mu+2-\varrho} \frac{\Gamma((\varrho+\mu)/2 - \mu - 1)}{\Gamma(\mu+2+(\nu-\varrho)/2)} {}_1F_2 \\ &\times \left(\mu+1; \mu+2+\frac{(\nu-\varrho)}{2}, \mu+2-\frac{(\nu+\varrho)}{2}, \frac{a^2}{4}\right) \end{aligned} \tag{155}$$

for which $a > 0$ and $\mathrm{Re}\,\nu < \mathrm{Re}\,\varrho < 2\mathrm{Re}\,\mu + 7/2$ is of use. Since the wavefunction is symmetric with respect to the interchange of the indices n and j_0, it is sufficient to restrict attention to the case $n \geqslant j_0$. Thus $a = \lambda t > 0$, $\varrho = 2\mathrm{j}+\mathrm{n}-\mathrm{j}_0+2$, $\mu = \mathrm{s}-\mathrm{i}k$, $\nu = n - j_0 \geqslant 0$, hence $-\mathrm{Re}\,\nu < \mathrm{Re}\,\varrho$. Moreover, since this integral is included in a sum for which the nonzero terms are governed by the non-vanishing of Δ, the constraints $s-n-j \geqslant 0$ and $s+j_0-j \geqslant 0$ need to be taken into account. They ensure that there are no negative arguments in the factorials. As all such conditions are satisfied in this development, the integral formula can be used in equation (154). First, only the asymptotic form of this formula, i.e. the limit $t \to 0$ is required. In this limit the hypergeometric functions equate to 1, and the overall normalization is irrelevant. Collecting everything

$$\begin{aligned} \lim_{t\to 0} \psi^{kj_0}_{\lambda sn}(t) &\sim \mathrm{i}^{n+j_0} \left(\frac{\lambda}{2}\right)^{\mathrm{i}k} t^{\mathrm{i}k} \left[\sum_{j=0}^{\infty} (-1)^{j+n-j_0} \Delta_{sj,nj_0} \left(\frac{\lambda t}{2}\right)^{n-j_0} \right. \\ &\left. \times \frac{\Gamma(j+n-j_0+1)\Gamma(s-j-\mathrm{i}k+j_0-n)}{\Gamma(s+1-\mathrm{i}k)\Gamma(n-j_0+1)}\right] + \mathrm{i}^{n+j_0}\left(\frac{\lambda}{2}\right)^{-\mathrm{i}k} t^{-\mathrm{i}k} \\ &\times \left[\sum_{j=0}^{\infty} (-1)^{j+n-j_0} \Delta_{sj,nj_0} \left(\frac{\lambda t}{2}\right)^{2s-2j-(n-j_0)} \frac{\Gamma(j-s+\mathrm{i}k+n-j_0)}{\Gamma(s+1-\mathrm{i}k-j)}\right]. \end{aligned} \tag{156}$$

Since $n \geqslant j_0$, $t^{n-j_0} = \mathrm{e}^{-r(n-j_0)}$ vanishes exponentially unless $n = j_0$. Hence to get the correct asymptotic behaviour with the first term, n must be equal to j_0. In the second term an exponential decay occurs unless $j = s-(n-j_0)/2$. With the constraints $s-n-j \geqslant 0$ and $s+j_0-j \geqslant 0$ for $\Delta \neq 0$, these enforce the constraint $n = -j_0$. Thus there is the asymptotic limit

$$\begin{aligned} \lim_{r\to\infty} \psi^{kj_0}_{\lambda ns}(r) &\sim \left(\frac{\lambda}{2}\right)^{\mathrm{i}k} \mathrm{i}^{2m} \left[\sum_{j=0}^{\infty} (-1)^j \Delta_{sj,nn} B(j+1, s-j-\mathrm{i}k)\right] \mathrm{e}^{-\mathrm{i}kr} \delta_{n,j_0} \\ &+ \left(\frac{\lambda}{2}\right)^{-\mathrm{i}k} \left[(-1)^{s+n} \Delta_{ss-n,n-n} \frac{\Gamma(n+\mathrm{i}k)}{\Gamma(n+1-\mathrm{i}k)}\right] \mathrm{e}^{\mathrm{i}kr} \delta_{n,-j_0} \end{aligned} \tag{157}$$

where $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. Then as $\Delta_{ss-n,n-n}=1$ and

$$B(j+1,s-j-\mathrm{i}k)=\frac{\Gamma(j+1)\Gamma(s-j-\mathrm{i}k)}{\Gamma(1+s-\mathrm{i}k)}=\int_0^\infty\frac{t^j}{(1+t)^{s+1-\mathrm{i}k}}\,\mathrm{d}t \tag{158}$$

the asymptotic limit is

$$\lim_{r\to\infty}\psi^{kj_0}_{\lambda ns}(r)\sim\mathrm{i}^{2n}\left(\frac{\lambda}{2}\right)^{\mathrm{i}k}\left[\int_0^\infty\mathrm{d}t\,(1+t)^{-(s+1-\mathrm{i}k)}\sum_{j=0}^{\infty}\frac{(-s-n)_j(-s+n)_j}{(1)_j}\frac{(-t)^j}{j!}\right]\delta_{n,j_0}\,\mathrm{e}^{-\mathrm{i}kr}$$
$$+\left(\frac{\lambda}{2}\right)^{-\mathrm{i}k}(-1)^{s+n}\frac{\Gamma(n+\mathrm{i}k)}{\Gamma(n+1-\mathrm{i}k)}\delta_{n,-j_0}\,\mathrm{e}^{\mathrm{i}kr}. \tag{159}$$

To calculate the scattering matrix from this, the integral of the (finite) sum must be done. It can be noted that the sum is precisely the hypergeometric function ${}_2F_1(-s-n,-s+n;1;-t)$. Hence the integral to be evaluated is

$$I(s,n,k)\equiv\int_0^\infty\mathrm{d}t(1+t)^{-(s+1-\mathrm{i}k)}{}_2F_1(-s-n,-s+n;1;-t) \tag{160}$$

and using [23] (p 854), namely,

$$\int_0^\infty\mathrm{d}t\,t^{\gamma-1}(z+t)^{-\sigma}{}_2F_1(\alpha,\beta;\gamma;-t)=\frac{\Gamma(\gamma)\Gamma(\alpha-\gamma+\sigma)\Gamma(\beta-\gamma+\sigma)}{\Gamma(\sigma)\Gamma(\alpha+\beta-\gamma+\sigma)}{}_2F_1$$
$$\times(\alpha-\gamma+\sigma,\beta-\gamma+\sigma;\alpha+\beta-\gamma+\sigma;1-z) \tag{161}$$

where $\mathrm{Re}\,\gamma>0$, $\mathrm{Re}(\alpha-\gamma+\sigma)>0$, $\mathrm{Re}(\beta-\gamma+\sigma)>0$ and $|\arg z|<\pi$, with $\gamma=1$, $z=1$, the integral can be expressed entirely in terms of gamma functions, i.e.

$$I(s,n,k)=\frac{\Gamma(-n-\mathrm{i}k)\Gamma(n-\mathrm{i}k)}{\Gamma(s+1-\mathrm{i}k)\Gamma(-s-\mathrm{i}k)}=(-1)^{s+n}\frac{\Gamma(n-\mathrm{i}k)\Gamma(s+1+\mathrm{i}k)}{\Gamma(n+1+\mathrm{i}k)\Gamma(s+1-\mathrm{i}k)}. \tag{162}$$

Note that the reflection formula, $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$, has been used as well. As a result, the scattering matrix is given by

$$\mathcal{S}_{nn'}(\lambda,k,s)=\left(\frac{\lambda}{2}\right)^{-2\mathrm{i}k}\frac{\Gamma(n+\mathrm{i}k)}{\Gamma(n-\mathrm{i}k)}\frac{\Gamma(n+1+\mathrm{i}k)}{\Gamma(n+1-\mathrm{i}k)}\frac{\Gamma(s+1-\mathrm{i}k)}{\Gamma(s+1+\mathrm{i}k)}\mathrm{i}^{2n}\delta_{n,-n} \tag{163}$$

where $-s\leqslant n,n'\leqslant s$.

Finally, in this appendix, I prove for the spin-$\frac{1}{2}$ case that the representation (52) of the coupled channel wavefunction equates to that used in section 5. For spin $\frac{1}{2}$ because of symmetric structure, only three cases $n=j_0=\frac{1}{2}$, $n=j_0=-\frac{1}{2}$ and $n=-j_0=\frac{1}{2}$ remain operative. Using equations (153) and (154), the wavefunction when multiplied by $\sqrt{q}$ is

$$\sqrt{q}\psi^{j_0}_n(q)=N^k_{\lambda s}\frac{\mathrm{i}^{n+j_0}(-1)^{n-j_0}}{2\Gamma(3/2-\mathrm{i}k)}\left(\frac{\lambda}{2}\right)^{-\mathrm{i}k}(Q(n-j_0+\mathrm{i}k)+Q(1-n+j_0-\mathrm{i}k)) \tag{164}$$

where

$$Q(x)\equiv2^{-2x}q^{1/2+x}{}_0F_1(;x+1/2;(q/4)^2)\Gamma(1/2-x) \tag{165}$$

and $q=2\lambda\,\mathrm{e}^{-r}$, and the fact that ${}_1F_2(c;a+1/2,c;z)={}_0F_1(;a+1/2;z)$ has been used. Then one can use the relation [24]

$${}_0F_1(;a+1/2;(q/4)^2)=\mathrm{e}^{-q/2}{}_1F_1(a;2a;q) \tag{166}$$

and equations (13.4.4) and (13.4.5) on page 506 of [17], namely,

$$\frac{q}{2a-1}{}_1F_1(a;2a;q)={}_1F_1(a;2a-1;q)-{}_1F_1(a-1;2a-1;q)$$
$$2{}_1F_1(a;2a;q)={}_1F_1(a;2a+1;q)+{}_1F_1(a+1;2a+1;q). \tag{167}$$

For $n = j_0$, the quantities $Q(\mathrm{i}k)$ and $Q(1-\mathrm{i}k)$ are required, and for $n = -j_0 = \frac{1}{2}$ it is $Q(-\mathrm{i}k)$ and $Q(1+\mathrm{i}k)$. Hence only the $n = j_0$ case needs to be resolved since the other follows on substitution of ik by $-\mathrm{i}k$. Using equations (74) and (167) it follows that

$$\begin{aligned} Q(\mathrm{i}k) &= \tfrac{1}{2}2^{-2\mathrm{i}k}\Gamma(1/2-\mathrm{i}k)\left[M_{-\frac{1}{2},\mathrm{i}k}(q)+M_{\frac{1}{2},\mathrm{i}k}(q)\right] \\ Q(1-\mathrm{i}k) &= \tfrac{1}{2}2^{2\mathrm{i}k}\Gamma(1/2+\mathrm{i}k)\left[M_{-\frac{1}{2},-\mathrm{i}k}(q)-M_{\frac{1}{2},-\mathrm{i}k}(q)\right]. \end{aligned} \tag{168}$$

Finally, using the duplication formula

$$2^{\pm 2\mathrm{i}k}\Gamma(1/2\pm\mathrm{i}k) = \sqrt{4\pi}\frac{\Gamma(\pm 2\mathrm{i}k)}{\Gamma(\pm\mathrm{i}k)} \tag{169}$$

one can find

$$\psi^k_{\lambda\frac{1}{2}}(q) = \mathcal{N}^k_{\lambda\frac{1}{2}}q^{-1/2}\mathcal{R}^k_{\lambda\frac{1}{2}}(q) \tag{170}$$

where $\mathcal{R}(q)$ is given in the text and the normalization factor is

$$\mathcal{N}^k_{\lambda\frac{1}{2}} \equiv -\sqrt{\left(\frac{\lambda\pi\cos\pi(\lambda/2+\mathrm{i}k)}{1/2+2k^2}\right)}. \tag{171}$$

Appendix C

In this appendix I outline how one can find the eigenvectors of $S_2 = \frac{1}{2\mathrm{i}}(S_+ - S_-)$. Recall that

$$S_\pm|s\mu\rangle = \sqrt{(s\pm\mu+1)(s\mp\mu)}|s\mu\pm 1\rangle \tag{172}$$

and write the eigenvalue problem in the form

$$S_2 u_m(\mu) = (s-m)u_m(\mu) \qquad m = 0,1,\ldots,2s \quad -s\leqslant\mu\leqslant s. \tag{173}$$

With normalized eigenvectors in the $|s\mu\rangle$ base defined by $u_m(\mu) = N_{sm}Q_m(\mu)$, N_{sm} is a normalization factor, and with $Q_m(\mu)$ having the form

$$Q_m(\mu) = \mathrm{i}^\mu\sqrt{\omega(\mu)}P_m(\mu) \qquad \omega^{-1}(\mu) = (s-\mu)!(s+\mu)! \tag{174}$$

one can show that $P_m(\mu)$ satisfies the equation

$$(s-\mu)P_m(\mu+1)+(s+\mu)P_m(\mu-1) = 2(s-m)P_m(\mu). \tag{175}$$

It is known [18] (see p 349) that the Krawtchouk polynomials satisfy the recursion relation

$$p(N-x)K_s(x+1;p;N)+(1-p)xK_s(x-1;p;N) = (p(N-2x)+x-s)K_s(x;p;N) \tag{176}$$

which is precisely equation (175) with $N = 2s, x = s+\mu, s = m$ and $p = \frac{1}{2}$. Hence I conclude that

$$Q_m(\mu) = \frac{\mathrm{i}^\mu}{\sqrt{(s-\mu)!(s+\mu)!}}K_m\left(s+\mu;\frac{1}{2};2s\right). \tag{177}$$

To find the normalization factor formula (9) in [18], page 348,

$$\sum_{x=0}^{N} C_N^x p^x(1-p)^{N-x}K_s(x;p;N)K_q(x;p;N) = \left(\frac{1-p}{p}\right)^s\left(C_N^s\right)^{-1}\delta_{sq} \quad C_N^x \equiv \binom{N}{x} \tag{178}$$

is useful when seeking a relation $\sum_{\mu=-s}^{s} \overline{u_m}(\mu) u_{m'}(\mu) = \delta_{mm'}$. It is straightforward to find then that the normalized eigenvectors of S_2 are

$$u_{\mu\alpha} = \frac{\mathrm{i}^{\mu} 2^{-s}(2s)!}{\sqrt{(s+\mu)!(s-\mu)!(s+\alpha)!(s-\alpha)!}} K_{s+\alpha}\left(s+\mu; \frac{1}{2}; 2s\right) \tag{179}$$

where the Krawtchouk polynomials can be generated by [19]

$$K_{s+\alpha}\left(s+\mu; \frac{1}{2}; 2s\right) = \frac{(s+\mu)!(s-\mu)!(s-\alpha)!}{(2s)!} \Delta^{s+\alpha} \left[\frac{1}{(s+\mu)!(s-\mu)!} \prod_{j=0}^{s+\alpha-1} (s+\mu-j)\right] \tag{180}$$

with the definition

$$\Delta F(\mu) \equiv F(\mu+1) - F(\mu). \tag{181}$$

Note also that an instructive notation, $u_{\mu\alpha} \equiv u_{s+\alpha}(\mu)$, has been used. That is, here α is that index used in $m = s + \alpha$. $u_{\mu\alpha}$ is a $(2s+1) \times (2s+1)$-matrix with columns labelled by $-s \leqslant \alpha \leqslant s$ corresponding to the different normalized eigenvectors linked to the eigenvalue α. The subscript μ labels the different components of the eigenvector in question.

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